MAT 1100, Algebra I, Fall 2016
Homework 3, due on Thursday, November 3

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1. Show the following are equivalent:

   (a) Every finite group of odd order is solvable.

   (b) Every non-abelian finite simple group is of even order.

   Remark: a famous theorem of Feit and Thompson (1963) shows that (a) is true.

2. (a) If $|G| = pq$ with $p < q$ both prime, show that G is solvable.

   (b) If $|G| = pqr$ with $p < q < r$ all prime, show that G is solvable. (Hint: if G is simple, give a lower bound for n_p, n_q, n_r and hence for the number of elements of order p, q, r. Show that their sum is greater than $|G|$.)

3. Suppose that G is a finite solvable group. Let $M \lhd G$ be a *minimal* normal subgroup (i.e. $M \neq 1$, and if $N \lhd G$, $N \leq M$, then $N = 1$ or $N = M$). Show that M is abelian and that there exists a prime number p such that every non-identity element of M has order p. (In fact, we will see later that this implies $M \cong (\mathbb{Z}/p)^r$ for some r.)

   (Hint: use characteristic subgroups of M to deduce first that M is abelian, then that M is a p-group, finally the claim.)

4. Identify the following (familiar) group: $\langle x, y \mid yxy^{-1} = x^{-1}, xyx^{-1} = y^{-1}\rangle$. (Hint: first try to establish more relations, e.g. $x^4 = 1$.)

5. (a) Consider the group $G = \langle x, y \mid x^2 = y^2 = 1\rangle$. Show that G is isomorphic to a subgroup H of $S_{\mathbb{Z}}$, the group of permutations of the set $\mathbb{Z}$. (Hint: send x to $a \mapsto -a$ and y to $a \mapsto 1 - a$.) Show that $H = \mathbb{Z} \rtimes \mathrm{Aut}(\mathbb{Z})$, an internal semidirect product, where $\mathbb{Z}$ is the subgroup of translations and $\mathrm{Aut}(\mathbb{Z})$ the automorphism group of $\mathbb{Z}$ (of order 2).

   (b) Let G be a group and N a normal subgroup of G such that $G/N \cong F(S)$ for some set S. Prove that $G \cong N \rtimes_\theta F(S)$ for some group homomorphism $\theta : F(S) \to \mathrm{Aut}\, N$.

6. Suppose that $G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ with $m < n$. The purpose of this exercise is to show that G is infinite. (Note that this can fail if $m = n$.)

   (a) Show that it suffices to prove the existence of a *non-zero* homomorphism $G \to \mathbb{Z}$.

   (b) As a warmup, use the universal property of $\langle S \mid R \rangle$ to construct a non-zero homomorphism $\langle x, y \mid xy^3x^4y^{-1} \rangle \to \mathbb{Z}$. In fact, find all possible such homomorphisms.

   (c) Now show that in general there is a non-zero homomorphism $G \to \mathbb{Z}$. It may help to use facts from linear algebra...