MAT 1100, Algebra I, Fall 2015
Homework 3, due on Friday November 6
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1. (a) If $|G| = pq$ with $p < q$ both prime, show that G is solvable.

   (b) If $|G| = pqr$ with $p < q < r$ all prime, show that G is solvable. (Hint: if G is simple, give a lower bound for n_p, n_q, n_r and hence for the number of elements of order p, q, r. Show that their sum is greater than $|G|$.)

2. Show that the subgroup $B \leq \mathrm{GL}_n(\mathbb{F}_p)$ consisting of all upper-triangular matrices is solvable. (Hint: it may be useful to consider a normal p-subgroup.)

3. (a) Show that any group G is a quotient group of some free group.

   (b) Show that any group G admits a presentation, i.e. $G \cong \langle S \mid R \rangle$ for some set S and some set of words R (in S).

4. Identify the following (familiar) group: $\langle x, y \mid yxy^{-1} = x^{-1}, xyx^{-1} = y^{-1} \rangle$. (Hint: first try to establish more relations, e.g. $x^4 = 1$.)

5. Consider the group $G = \langle s, t \mid s^2 = t^3 = (st)^3 = 1 \rangle$.

   (a) Show that G has at most 12 elements. (Hint: show that each element is represented by a word of length at most 3.)

   (b) Using (a), conclude that $G \cong A_4$.

6. Recall that a free group $F(S)$ on a set S comes equipped with a function $\alpha_S : S \to F(S)$. Also recall that for any function $\beta : S \to G$, for some group G, there exists a unique homomorphism $\widetilde{\beta} : F(S) \to G$ such that $\widetilde{\beta} \circ \alpha_S = \beta$.

   (a) Suppose that $f : S \to T$ is a function between sets S, T. Show that there exists a unique homomorphism between free groups $F(f) : F(S) \to F(T)$ such that $F(f) \circ \alpha_S = \alpha_T \circ f$.

   (b) Show that we have $F(f \circ g) = F(f) \circ F(g)$ (whenever f, g are functions such that the target of g is the domain of f) and $F(1_S) = 1_{F(S)}$.

(c) Show that for any functions $\beta : S \to G$, $f : T \to S$ and homomorphism $\phi : G \to H$ we have:

$$\widetilde{\beta \circ f} = \widetilde{\beta} \circ F(f); \quad \widetilde{\phi \circ \beta} = \phi \circ \widetilde{\beta}.$$