MAT 1100, Algebra I, Fall 2019
Homework 2, due on Tuesday, October 15
Florian Herzig

1. A subgroup H of a group G is called characteristic (in G) if for all automorphisms φ of G one has $\varphi(H) = H$.
   (a) Show that the centre $Z(G)$ is characteristic in G.
   (b) Prove that any characteristic subgroup of a group is normal.
   (c) Give an example of a group and a normal subgroup that is not characteristic.
   (d) Show that if $K \leq H \leq G$ and K is characteristic in H and H normal in G, then K is normal in G.
   (e) Show that the relation of being a characteristic subgroup is transitive. What about the relation of being a normal subgroup? (Give a proof or counterexample.)

2. Suppose that G is a finite group
   (a) If $N \lhd G$ and P a Sylow subgroup of N, prove that $G = N_G(P)N$.
   (b) Suppose that every maximal subgroup of G is normal in G. Prove that every Sylow subgroup P of G is normal in G. Here, a *maximal subgroup* is a proper subgroup that is maximal among all proper subgroups. For example, any subgroup of prime index is maximal. (Hint: if P is not normal, its normaliser is contained in some maximal subgroup. . .)

3. Suppose that $N \lhd G$, so we say that $1 \to N \xrightarrow{i} G \xrightarrow{\pi} G/N \to 1$ is a short exact sequence.
   (a) If we have a *splitting* (or *section*) $s : G/N \to G$, i.e. a homomorphism s such that $\pi \circ s = \mathrm{id}_{G/N}$, then $G = N \rtimes \mathrm{im}(s)$, an internal semidirect product, and $\mathrm{im}(s) \cong G/N$.
   (b) If we have a *splitting* (or *retraction*) $r : G \to N$, i.e. a homomorphism r such that $r \circ i = \mathrm{id}_N$, then $G = N \times \ker(r)$, an internal direct product, and $\ker(r) \cong G/N$.

(c) *Use part (a)* to exhibit $\mathrm{GL}_n(k)$ as a semidirect product of $\mathrm{SL}_n(k)$ and $k^\times$, where k is any field.

4. Suppose that X_1, X_2 are two G-sets (with G-actions denoted by $\cdot_1$ and $\cdot_2$, respectively). We say that a *map of G-sets* is a function $f : X_1 \to X_2$ such that $f(g \cdot_1 x) = g \cdot_2 f(x)$ for all $g \in G$, $x \in X_1$. We say that f is an *isomorphism of G-sets* if f is moreover a bijection.

   (a) Show that any transitive G-set is isomorphic to G/H (with G acting by left multiplication) for some subgroup $H \leq G$.

   (b) Given $H_1, H_2 \leq G$ show that there exists a map of G-sets $f : G/H_1 \to G/H_2$ if and only if H_1 is contained in some conjugate of H_2.

   (c) Show that $G/H_1 \cong G/H_2$ as G-sets if and only if H_1 and H_2 are conjugate subgroups.

5. (a) Suppose that H and N are groups and that $\phi : H \to \operatorname{Aut} N$ is a homomorphism. For any $\sigma \in \operatorname{Aut}(H)$ exhibit an explicit isomorphism $N \rtimes_{\phi\circ\sigma} H \cong N \rtimes_\phi H$.

   (b) If $p < q$ are primes such that p divides $q - 1$, prove that up to isomorphism there are precisely two groups of order pq (one abelian and one nonabelian). You may use that the group $(\mathbb{Z}/q)^\times$ is cyclic.

   (c) Use a semidirect product to construct a nonabelian group of order 56 such that all its Sylow subgroups are cyclic.