MAT 1100, Algebra I, Fall 2016
Homework 2, due on Wednesday, October 12
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1. A subgroup H of a group G is called characteristic (in G) if for all automorphisms φ of G one has $\varphi(H) = H$.

   (a) Show that the centre $Z(G)$ is characteristic in G.

   (b) Prove that any characteristic subgroup of a group is normal.

   (c) Give an example of a group and a normal subgroup that is not characteristic.

   (d) Show that if $K \leq H \leq G$ and K is characteristic in H and H normal in G, then K is normal in G.

   (e) Show that the relation of being a characteristic subgroup is transitive. What about the relation of being a normal subgroup?

2. Suppose that G ($\neq 1$) is finite and that $H \leq G$ such that $(G : H) = p$ is the *smallest* prime number dividing $|G|$.

   (a) Let $X := \{gHg^{-1} : g \in G\}$. Show that $|X| \in \{1, p\}$.

   (b) Show that $H \lhd G$. (Hint: if $|X| = p$ consider the kernel of a suitable action $G \to S_X$.)

3. Suppose that G is a finite group.

   (a) If P is a Sylow p-subgroup, show that $N_G(N_G(P)) = N_G(P)$.

   (b) Suppose that H is any p-subgroup of G such that the index $(G : H)$ is divisible by p. Show that $(N_G(H) : H)$ is divisible by p. (Hint: consider the action of H on G/H. If you can't do the problem, perhaps you can at least show that $H \neq N_G(H)$?)

   (c) Deduce that if G is a p-group and $H \lneq G$ is a proper subgroup, then $H \neq N_G(H)$.

4. Prove that S_4 is isomorphic to $V_4 \rtimes_\psi S_3$ with respect to some *isomorphism* $\psi : S_3 \to \operatorname{Aut} V_4$. Here, V_4 denotes the Klein 4-group, i.e. $V_4 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$.

5. Prove Proposition 30 from class: suppose that $N \lhd G$, so we say that $1 \to N \xrightarrow{i} G \xrightarrow{\pi} G/N \to 1$ is a short exact sequence.

   (a) If we have a *splitting* (or *section*) $s : G/N \to G$, i.e. a homomorphism s such that $\pi \circ s = \mathrm{id}_{G/N}$, then $G = N \rtimes \mathrm{im}(s)$, an internal semidirect product, and $\mathrm{im}(s) \cong G/N$.

   (b) If we have a *splitting* (or *retraction*) $r : G \to N$, i.e. a homomorphism r such that $r \circ i = \mathrm{id}_N$, then $G = N \times \ker(r)$, an internal direct product, and $\ker(r) \cong G/N$.