MAT 1100, Algebra I, Fall 2015
Homework 2, due on Friday, October 16[1]

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1. (a) Suppose that $H \leq G$. Show that $N := \bigcap_{g \in G} gHg^{-1}$ is the largest normal subgroup of G that is contained in H. (In other words, $N \lhd G$, $N \leq H$, and if M is a subgroup of G with $M \lhd G$, $M \leq H$, then $M \leq N$.)

   (b) Suppose that $(G : H)$ is finite. Show that there exists a normal subgroup $N \lhd G$ with $N \leq H$ such that $(G : N)$ is finite. (Hint: consider a suitable G-action $G \to S_X$.) Conclude that for any $g \in G$ there exists $n \geq 1$ such that $g^n \in H$.

   (c) Suppose that G is finite. Show that the intersection of all Sylow p-subgroups of G is the largest normal p-subgroup of G.

2. Suppose that G $(\neq 1)$ is finite and that $H \leq G$ such that $(G : H) = p$ is the *smallest* prime number dividing $|G|$.

   (a) Let $X := \{gHg^{-1} : g \in G\}$. Show that $|X| \in \{1, p\}$.

   (b) Show that $H \lhd G$. (Hint: if $|X| = p$ consider the kernel of a suitable action $G \to S_X$.)

3. Prove Proposition 30 from class: suppose that $N \lhd G$, so we say that $1 \to N \xrightarrow{i} G \xrightarrow{\pi} G/N \to 1$ is a short exact sequence.

   (a) If we have a *splitting* (or *section*) $s : G/N \to G$, i.e. a homomorphism s such that $\pi \circ s = \mathrm{id}_{G/N}$, then $G = N \rtimes \mathrm{im}(s)$, an internal semidirect product, and $\mathrm{im}(s) \cong G/N$.

   (b) If we have a *splitting* (or *retraction*) $r : G \to N$, i.e. a homomorphism r such that $r \circ i = \mathrm{id}_N$, then $G = N \times \ker(r)$, an internal direct product, and $\ker(r) \cong G/N$.

4. (a) Show that Q_8 isn't an internal semidirect product of two nontrivial subgroups. (Hint: show that Q_8 contains a nontrivial element that is contained in every nontrivial subgroup.)

[1] You can also hand it in on Monday, October 19, but in that case I cannot guarantee that it will be graded before the term test.

(b) Show that $\mathrm{Aut}(\mathbb{Z}/5)$ is cyclic of order 4, and use this to construct a nonabelian group of order 20.

5. Suppose $N \lhd G$. In the context of the correspondence theorem (as stated in Theorem 9 in class) show that if subgroups $\overline{H}_1 \le \overline{H}_2 \le G/N$ correspond to subgroups $H_1 \le H_2 \le G$ containing N, then $H_1 \lhd H_2$ iff $\overline{H}_1 \lhd \overline{H}_2$, and if this holds, then $H_2/H_1 \cong \overline{H}_2/\overline{H}_1$.

6. Suppose that X_1, X_2 are two G-sets (with G-actions denoted by $\cdot_1$ and $\cdot_2$, respectively). We say that a *morphism of G-sets* is a function $f : X_1 \to X_2$ such that $f(g \cdot_1 x) = g \cdot_2 f(x)$ for all $g \in G$, $x \in X_1$. We say that f is an *isomorphism of G-sets* if f is moreover a bijection.

   (a) Show that any transitive G-set is isomorphic to G/H (with G acting by left multiplication) for some subgroup $H \le G$.

   (b) Given $H_1, H_2 \le G$ show that there exists a morphism of G-sets $f : G/H_1 \to G/H_2$ if and only if H_1 is contained in some conjugate of H_2.

   (c) Show that $G/H_1 \cong G/H_2$ as G-sets if and only if H_1 and H_2 are conjugate subgroups.