MAT 1100, Algebra I, Fall 2019
Homework 1, due on Tuesday October 1

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1. Suppose that G is a group with subgroups H, K. Show that $HK = \{hk : h \in H, k \in K\}$ is a subgroup of G if and only if $HK = KH$.

2. Suppose that G is a group.

   (a) An *inner* automorphism of G is an automorphism that is induced by conjugation by some $x \in G$. Show that the inner automorphisms $\mathrm{Inn}(G)$ form a subgroup of the automorphism group $\mathrm{Aut}(G)$ of G.

   (b) Let $Z(G) := \{g \in G : gh = hg \ \forall h \in G\}$ (the *centre of* G). Show that any subgroup of $Z(G)$ is normal in G.

   (c) Show that $\mathrm{Inn}(G) \cong G/Z(G)$ and that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

   (d) Give an example, with proof, of an outer automorphism of a finite group. (An outer automorphism is one that is not inner.)

   (e) Show that if $G/Z(G)$ is cyclic, then G is abelian.

3. Suppose that $\sigma \in S_n$ is any element, with cycle type $(n_1, n_2, \ldots, n_r)$ (so $n = n_1 + \cdots + n_r$).

   (a) Express the order of σ in terms of the integers n_i.

   (b) Find the smallest n such that some element of S_n has order 14. Give an example of such an element.

4. Suppose that G is a finite group and N a normal subgroup such that $|N|$, $|G/N|$ are relatively prime. Show that N is the *only* subgroup of G of order $|N|$. (Hint: suppose that H is another subgroup of that order and consider $HN\ldots$)

5. We want to classify nonabelian groups of order 8.

   (a) Show that if G is any group such that $x^2 = 1$ for all $x \in G$, then G is abelian.

(b) Suppose from now on that G is a nonabelian group of order 8. Use (a) to show that G contains an element g of order 4. Deduce that G contains a normal subgroup N that is cyclic of order 4.

(c) Let x denote any element of G that is not in N. Show that $xgx^{-1} = g^{-1}$. (Use that G is nonabelian.)

(d) If x has order 2, deduce that $G \cong D_8$.

(e) If x has order 4, show that any element of $G \setminus N$ has order 4 and hence there is a unique element of order 2. Deduce that $x^2 = g^2$ and show that the multiplication table of G is uniquely determined.

(f) Deduce that there is, up to isomorphism, a unique nonabelian group of order 8 that has precisely one element of order 2. Deduce that it is Q_8.

(g) Consider the subgroup $\left\{ \begin{pmatrix} 1 & a & b \\ & 1 & c \\ & & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_2 \right\}$ of $\mathrm{GL}_3(\mathbb{F}_2)$. Which group is it isomorphic to?

6. Let G be a finite group that has an automorphism σ such that $\sigma^2 = 1$ and such that $\sigma(g) = g$ if and only if $g = 1$. Prove that G is abelian. (Hint: show that every element of G is of the form $x^{-1}\sigma(x)$.)