MAT 1100, Algebra I, Fall 2016
Homework 1, due on Tuesday October 4

Florian Herzig

1. (a) Let G be a group in which $g^2 = 1$ for every $g \in G$. Show that G is abelian.

   (b) Let G be a finite group whose only subgroups are $\{1\}$ and G. Prove that either $G = \{1\}$ or that G is cyclic of prime order.

2. Suppose that G is a group.

   (a) Recall that an inner automorphism of G is an automorphism that is induced by conjugation by some $x \in G$. Show that the inner automorphisms $\mathrm{Inn}(G)$ form a subgroup of the automorphism group $\mathrm{Aut}(G)$ of G.

   (b) Let $Z(G) := \{g \in G : gh = hg \ \forall h \in G\}$ (the *centre of* G). Show that any subgroup of $Z(G)$ is normal in G.

   (c) Show that $\mathrm{Inn}(G) \cong G/Z(G)$ and that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

   (d) Give an example, with proof, of an outer automorphism of a finite group. (An outer automorphism is one that is not inner.)

3. Suppose that G is a finite group and N a normal subgroup such that $|N|$, $|G/N|$ are relatively prime. Show that N is the *only* subgroup of G of order $|N|$. (Hint: suppose that H is another subgroup of that order and consider $HN\dots$)

4. (a) Let G, H be finite groups and $\phi : G \to H$ homomorphisms. Show that the order of the image, $|\operatorname{im}(\phi)|$, divides $\gcd(|G|, |H|)$.

   (b) Suppose that $(G, +)$ is an abelian group and $n \geq 1$. Let $G[n] := \{x \in G : nx = 0\}$. (Here we think of G with additive notation, so 0 is the identity and $nx = x + \cdots + x$, n times.) Show that $G/G[n]$ is isomorphic to a subgroup of G. Identify this subgroup.

5. Let G be a group, $N \lhd G$ a normal subgroup, and X a G-set.

   (a) Prove that there are unique actions of G on X^N and on $N \backslash X$ for which the inclusion map $X^N \hookrightarrow X$ and the natural map $X \to$

$N\backslash X$ are G-maps. Show also that these actions are induced by actions of G/N on X^N and on $N\backslash X$, via the canonical map $G \to G/N$. (Here, $X^N := \{x \in X : nx = x \ \forall n \in N\}$ and $N\backslash X$ denotes the set of orbits of N on X.)

(b) Suppose that $H \leq G$ is a subgroup, and let X be the G-set G/H. Suppose that there is a G-action on X^H for which the inclusion $X^H \hookrightarrow X$ is a G-map, or that there is a G-action on $H\backslash X$ for which the natural map $X \to H\backslash X$ is a G-map. Prove that H is normal in G.

6. (a) Show that there is no simple group of order pq, where p, q are distinct primes.

(b) Show that there is no simple group of order p^2q, where p, q are distinct primes. (Hint: use Sylow to show that $n_p = q$, $n_q = p^2$, … Rule out $(p, q) = (2, 3)$ e.g. by counting elements of order 3.)