MAT 1100, Algebra I, Fall 2015
Homework 1, due on Friday October 9
Florian Herzig

1. (a) Show that any group of order 4 is isomorphic to either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \times \mathbb{Z}/2$. (Note that I abbreviate $\mathbb{Z}/n\mathbb{Z}$ as $\mathbb{Z}/n$.)

   (b) Show that any group of order at most 5 is abelian.

2. Consider the following two groups of order 8: the dihedral group D_8 and the quaternion group $Q_8 := \{\pm 1, \pm i, \pm j, \pm k\}$ (a subset of Hamilton's quaternions with usual rules $i^2 = j^2 = k^2 = ijk = -1$).

   (a) Show that D_8 and Q_8 are non-abelian.

   (b) Show that $D_8 \not\cong Q_8$.

   (Fact: these are the only non-abelian groups of order 8.)

3. Let G be a group and let $Z(G) := \{g \in G : gh = hg \ \forall h \in G\}$ (the *centre of* G).

   (a) Show that $Z(G)$ is a normal subgroup of G. Better, can you find a "nice" homomorphism from G to another group whose kernel is $Z(G)$?

   (b) Give an example of a non-trivial group G such that $Z(G) = 1$.

   (c) Suppose that $G/Z(G)$ is cyclic. Show that G is abelian.

   (d) Give an example of a *non-abelian* group G such that $G/Z(G)$ is abelian. (Include your computation of $Z(G)$.)

4. Consider a cube in $\mathbb{R}^3$. The aim of this exercise is to show that the group Σ of rotational symmetries (i.e., the order-preserving isometries) of the cube is isomorphic to the symmetric group S_4.

   (a) By considering the action of Σ on the set of space diagonals of the cube construct a group homomorphism $\alpha : \Sigma \to S_4$.

   (b) Show that α is injective by showing geometrically why any element of the kernel has to be the identity.

(c) Show that α is surjective by exhibiting explicit elements of Σ (specifying the axis of rotation for each one) whose images under α generate S_4.

(d) Show that $|\Sigma| = 24$ by using the orbit-stabiliser theorem for the action of Σ on vertices or edges or faces (your choice!).

Please note: any two of (b), (c), (d) imply the third. I want you to solve all three of them independently without using such implications.

5. Suppose that G is a finite group and N a normal subgroup such that $|N|$, $|G/N|$ are relatively prime. Show that N is the *only* subgroup of G of order $|N|$. (Hint: suppose that H is another subgroup of that order and consider $HN \dots$)

6. (a) Show that there is no simple group of order pq, where p, q are distinct primes.

   (b) Show that there is no simple group of order p^2q, where p, q are distinct primes. (Hint: use Sylow to show that $n_p = q$, $n_q = p^2$, $\dots$ Rule out $(p, q) = (2, 3)$ e.g. by counting elements of order 3.)