Perturbative Aspects of the Chern-Simons Topological Quantum Field Theory

Dror Bar-Natan, June 1991

Nicely enough, the file containing the first few pages of my thesis was corrupted. These pages contained a nicer front page, a copyright notice, a nice Random Dot Streogram (see [6]), the abstract, and a part of the acknowledgement. So the following page begins right in the middle of the acknowledgement; I apologize.

evident throughout my thesis. Among them are Dan Abramovich, Scott Axelrod, Joan Birman, John Conway, Elias Stein, Michael Mintchev, Xiao-Song Lin, Carlos Simpson and Avraham Soffer. Dan Freed, William Thurston and Gerald Washnitzer were patient enough to listen to me, and I am grateful to them for that. I also owe special thanks to Professor Wu-Chung Hsiang who advised me in my first two years in Princeton.

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Thanks!

Contents

Chapter 1

The basic idea

1.1 The Chern-Simons path integral

The aim of this thesis is to explain some techniques originally developed by physicists studying quantum field theory, and to show how they can be used to derive three manifold and knot invariants. The basic idea is simple and to make it even simpler we will ignore knots for a moment and explain it first for the case of a bare three manifold. Our invariants will be complex numbers. To get a complex number out of a bare three manifold, that has no additional structure on it, is hard. It is a lot easier to get numerical quantities when there is more structure to play with. So we look at a three manifold with an additional piece of structure, generate a complex number using this additional structure, and then try to integrate our complex number over all possible choices of such an additional structure. The additional structure that we will pick will be a connection on some pre-picked bundle[1] on an oriented three manifold M^3, and the complex number that we will generate, the integrand in our program, will essentially be the exponential of the 'Lagrangian' — the Chern-Simons number [17] associated with the connection A:

$$cs(A) = \frac{1}{4\pi} \int_{M^3} \mathfrak{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

and so our invariant will be[2]:

$$\mathcal{W}(M^3, k) = \int_{\mathcal{A}} \mathcal{D}A \; e^{\frac{ik}{4\pi} \int_{M^3} \mathfrak{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)}. \tag{1.1}$$

(k is an integer parameter whose importance for our purposes will be made clear shortly).

[1]Namely, a principal G-bundle for some Lie group G. We also assume that G comes equipped with a bilinear non-degenerate invariant form $\mathfrak{tr}$ on its Lie algebra $\mathcal{G}$.

[2]For historical reasons, such integrals over infinite dimensional spaces are called *path integrals*. For the origin of the name, check [22].

To incorporate a link $\mathcal{X} = \{X_\gamma\}_{\gamma=1}^{\Gamma}$ into the above picture, we have to pick a list $\{R_\gamma\}_{\gamma=1}^{\Gamma}$ of finite dimensional representations of $\mathcal{G}$, and supplement the integrand:

$$\mathcal{W}(M^3, \mathcal{X}, k) \overset{\mathrm{def}}{=} \left\langle \prod_{\gamma=1}^{\Gamma} \mathcal{O}_{X_\gamma, R_\gamma} \right\rangle = \int_{\mathcal{A}} \mathcal{D}A \prod_{\gamma=1}^{\Gamma} \mathcal{O}_{X_\gamma, R_\gamma}(A) e^{ik\,\mathrm{cs(A)}} \tag{1.2}$$

Where[3]

$$\begin{aligned} \mathcal{O}_{X,R}(A) &= tr_R \mathcal{P}\mathrm{exp}\left(\int ds \dot{X}^i(s) A_i(X(s))\right) = \dim R - \int ds \dot{X}^i(s) A_i^a(X(s)) R^\alpha_{a\alpha} \\ &+ \int_{s_1<s_2} ds_{1,2} \dot{X}^{i_1}(s_1) \dot{X}^{i_2}(s_2) A_{i_1}^{a_1}(X(s_1)) A_{i_2}^{a_2}(X(s_2)) R^{\alpha_1}_{a_1\alpha_2} R^{\alpha_2}_{a_2\alpha_1} - \cdots . \end{aligned} \tag{1.3}$$

(1.3) is, of course, just the trace of the holonomy of the connection A along X in the representation R, expanded in powers of the connection A.

1.2 Perturbation theory and Feynman diagrams

1.2.1 Introduction

Luckily, the space of all connections $\mathcal{A}$ is an affine space and so there should be a canonical choice for a measure on it — the Lesbegue measure. Unluckily, $\mathcal{A}$ is an infinite dimensional space and so that measure doesn't really exist. To go around this we will use perturbation theory techniques that were originally developed by physicists to be used in quantum field theory. Instead of attempting to calculate the integral (1.1) as it is, we will try to investigate its asymptotic behavior as $k/2\pi i \to \infty$. It will turn out that (assuming that infinite dimensional Lesbegue measures do exist) to determine this asymptotic behavior requires only evaluating finite dimensional integrals represented by so-called "Feynman diagrams", and therefore it is possible to *define* the asymptotic behavior of (1.1) to be given by those "Feynman diagrams", without ever giving meaning to the integral (1.1) itself. I will very briefly present these techniques here. Further information can be found in any quantum field theory textbook such as [36, 21, 29].

To illustrate the technique of Feynman diagrams, let us first look at a simpler finite dimensional analogue — let $\mathcal{L}$ be a smooth real-valued function with finitely many stationary points $\{x_i\}_{i=1}^{I}$ on Euclidean space $\mathbf{R}^N$ (a 'Morse' function), and let us try to understand the $k \to \infty$ asymptotics of:

$$\mathcal{Z}_k = \int_{\mathbf{R}^N} d^N x \, e^{ik\mathcal{L}}.$$

[3] In this formula, as throughout the rest of this thesis, the Einstein summation convention applies — there is an implicit summation over indices (such as i, a, α, ...) that are repeated twice. Also notice the difference between tr_V and $\mathfrak{tr}$ — in this thesis $\mathfrak{tr}$ will always refer to an invariant bilinear form on a Lie-algebra, while tr_V is just the usual matrix trace in End$[V]$. Many times the subscript V will be omitted and matrix traces will simply be denoted by tr.

Namely, we will try to find constants $\mathcal{W}_0$, $\mathcal{W}_1$, ... so that asymptotically

$$\mathcal{Z}_k = \int_{\mathbf{R}^N} d^N x \, e^{ik\mathcal{L}} \underset{k \to \infty}{\sim} \sum_{i=1}^{I} \mathcal{W}_0^{(i)} e^{ik\mathcal{L}(x_i)} \sum_{m=1}^{\infty} \frac{\mathcal{W}_m^{(i)}}{k^m}. \tag{1.4}$$

1.2.2 The stationary phase approximation

The first step, even before Feynman diagrams are introduced, is to use the stationary phase principle which says that to zeroth order in $1/k$, the large k behavior of $\int \exp ik\mathcal{L}$ is given by

$$\int_{\mathbf{R}^N} d^N x \, e^{ik\mathcal{L}} \underset{k \to \infty}{\sim} \sum_{i=1}^{I} \frac{e^{\frac{i\pi}{4} \text{sign } L(x_i)}}{\sqrt{(4\pi k)^N \, |\det L(x_i)|}} e^{ik\mathcal{L}(x_i)}. \tag{1.5}$$

Here $L(x_i)$ is the Hessian matrix of $\mathcal{L}$ at x_i. In other words, $\det L(x_i)$ is the determinant of the operator $L(x_i) : T\mathbf{R}^N_{x_i} \to T\mathbf{R}^N_{x_i}$ defining (using the Euclidean inner product) the quadratic approximation to $f(x)$ around x_i. $\text{sign}\, L(x_i)$ is the signature of that quadratic approximation, i.e. the difference between the number of positive and the number of negative eigenvalues of $L(x_i)$.

The intuitive justification of (1.5) is the following. When k is positive and large and x is not near a stationary point, $k\mathcal{L}$ varies very rapidly around x, $\exp ik\mathcal{L}$ oscillates very rapidly, and therefore the points near x contribute very little to $\int \exp ik\mathcal{L}$. If x is near the stationary point x_i, then in a coordinate system $\{\xi_n\}$ around x_i in which $L(x_i)$ is diagonal with eigenvalues $\{\lambda_n\}$ we can approximate

$$\mathcal{L}(x) \underset{x \sim x_i}{\sim} \mathcal{L}(x_i) + \sum_{n=1}^{N} \lambda_n \xi_n^2.$$

This means that the contribution to $\int \exp ik\mathcal{L}$ from the points near x_i can be approximated by

$$e^{ik\mathcal{L}(x_i)} \lim_{\epsilon \to 0} \int_{\mathbf{R}^N} d^N \xi \, \exp\left(\sum_{n=1}^{N} ik\lambda_n \xi_n^2 - \epsilon \xi_n^2 \right), \tag{1.6}$$

where the convergence factor $-\epsilon\xi_n^2$ was inserted to account for the cancellations arising from the rapid oscillations of the integrand for large ξ. Computing the Gaussian integral (1.6) and then taking the $\epsilon \to 0$ limit, we get

$$(1.6) = e^{ik\mathcal{L}(x_i)} \prod_{n=1}^{N} \frac{e^{\frac{i\pi}{4} \text{sign } \lambda_n}}{2\sqrt{\pi k |\lambda_n|}}.$$

Summing over the stationary points, this is exactly (1.5).

A rigorous and more complete treatment of the stationary phase principle can be found in section 7.7 of Hörmander's book [28].

1.2.3 Feynman diagrams

Having computed the k independent constant factor $\mathcal{Z}$ in (1.4), we will next try to understand the part of (1.4) that does depend on k. For simplicity, let us now assume that $\mathcal{L}$ has just a single stationary point on $\mathbf{R}^N$, that this point is the origin, that $\mathcal{L}(0) = 0$, and that $\mathcal{L}$ near 0 is given by the sum of a non-degenerate quadratic form and a cubic correction to it. Therefore, the integral whose large k asymptotic behavior we want to determine is:

$$\mathcal{Z}_k = \int_{\mathbf{R}^N} d^N x \, e^{ik(\frac{1}{2}\lambda_{ij}x^i x^j + \lambda_{ijk}x^i x^j x^k)}. \tag{1.7}$$

The general case is not any harder.

By a simple change of variables,

$$\vec{x} \to \vec{x}' = \sqrt{k}\vec{x}, \tag{1.8}$$

(suppressing primes)

$$\mathcal{Z}_k = k^{-N/2} \int_{\mathbf{R}^N} d^N x \, e^{i\frac{1}{2}\lambda_{ij}x^i x^j} e^{\frac{i}{\sqrt{k}}\lambda_{ijk}x^i x^j x^k} \tag{1.9}$$

$$= k^{-N/2} \int_{\mathbf{R}^N} d^N x \, e^{i\frac{1}{2}\lambda_{ij}x^i x^j} \sum_{m=0}^{\infty} \frac{i^m}{m! k^{m/2}} (\lambda_{ijk}x^i x^j x^k)^m. \tag{1.10}$$

And so the mth term in our asymptotic expansion will be given up to a multiplicative constant by:

$$\int_{\mathbf{R}^N} d^N x \, e^{i\frac{1}{2}\lambda_{ij}x^i x^j} (\lambda_{ijk}x^i x^j x^k)^m =$$

this is a simple Gaussian integral, which we can evaluate using standard methods:

$$= \left[(\lambda_{ijk} \frac{-i\partial}{\partial J_i} \frac{-i\partial}{\partial J_j} \frac{-i\partial}{\partial J_k})^m \int_{\mathbf{R}^N} d^N x \, e^{i\frac{1}{2}\lambda_{ij}x^i x^j + iJ_i x^i} \right]_{\vec{J}=0}$$

$$\propto \left[(\lambda_{ijk} \frac{-i\partial}{\partial J_i} \frac{-i\partial}{\partial J_j} \frac{-i\partial}{\partial J_k})^m e^{-i\frac{1}{2}\lambda^{ij}J_i J_j} \right]_{\vec{J}=0}, \tag{1.11}$$

where λ^{ij} is the inverse of λ_{ij}: $\lambda_{ij}\lambda^{jk} = \delta_i{}^k$.

Now there are no more integrations to perform. The expression that we obtained can clearly be expanded further. The result of applying a differential operator to an exponential is a polynomial times that same exponential, and as we are evaluating this polynomial at 0, we are interested in its constant term. If we apply the $3m$ differentiations in (1.11) one at a time and use Leibnitz' rule to separate the derivatives to 'those that act on the exponential' and 'those that act on the polynomial' we see that the two types of differentiations have to be paired together — each differentiation that acts on the exponential 'brings down' a factor J, and each differentiation that acts on the polynomial eliminates such a factor. Remembering from (1.11) that the

differentiations come in triples coupled by a λ_{ijk}, we can represent the $3m$ differentiations in (1.11) by m 'cubic' vertices, and every pairing that contributes to (1.11) can be represented by a way of connecting these $3m$ vertices to make a graph. The graphes that are created in this way are called *Feynman diagrams.* Each vertex in such a diagram contributes a factor λ_{ijk}, and each edge a factor λ^{ij} (coming from the exponent in (1.11)). In summary, to evaluate (1.11) we calculate a sum over all Feynman diagrams with m cubic vertices of order three where the contribution of each such diagram is a product of λ_{ijk}'s for each vertex and λ^{ij}'s for each arc.
Example The term with $m = 2$ will be computed as follows:

$$
\begin{aligned}
\mathcal{W}_2 &= \left[(\lambda_{ijk}\frac{-i\partial}{\partial J_i}\frac{-i\partial}{\partial J_j}\frac{-i\partial}{\partial J_k})^2 \int_{\mathbf{R}^N} d^N x\, e^{i\frac{1}{2}\lambda_{ij}x^i x^j + iJ_i x^i}\right]_{\vec{J}=0} \\
&= \left[(\lambda_{ijk}\frac{-i\partial}{\partial J_i}\frac{-i\partial}{\partial J_j}\frac{-i\partial}{\partial J_k})(\lambda_{i'j'k'}\frac{-i\partial}{\partial J_{i'}}\frac{-i\partial}{\partial J_{j'}}\frac{-i\partial}{\partial J_{k'}}) \int_{\mathbf{R}^N} d^N x\, e^{i\frac{1}{2}\lambda_{ij}x^i x^j + iJ_i x^i}\right]_{\vec{J}=0} \\
\text{“=”} \quad & 6\left(\lambda_{ijk}\overbrace{\partial_i}^{1}\overbrace{\partial_j}^{2}\overbrace{\partial_k}^{3}\right)\left(\lambda_{i'j'k'}\overbrace{\partial_{i'}}^{1}\overbrace{\partial_{j'}}^{2}\overbrace{\partial_{k'}}^{3}\right)e^{\cdots} \\
& +9\left(\lambda_{ijk}\overbrace{\partial_i}^{1}\overbrace{\partial_j}^{1}\overbrace{\partial_k}^{2}\right)\left(\lambda_{i'j'k'}\overbrace{\partial_{i'}}^{2}\overbrace{\partial_{j'}}^{3}\overbrace{\partial_{k'}}^{3}\right)e^{\cdots} \\
&= (num1)\lambda_{ijk}\lambda_{i'j'k'}\lambda^{ii'}\lambda^{jj'}\lambda^{kk'} + (num2)\lambda_{ijk}\lambda_{i'j'k'}\lambda^{ij}\lambda^{kk'}\lambda^{i'j'}.
\end{aligned}
$$

The pairings in the last equation are represented by the following diagrams:

It is not hard to see that in general m is also equal to the number of independent loops in a diagram. Therefore we will also call the m'th order term in such an asymptotic expansion *the m-loop term.* It is customary to call the arcs of a Feynman diagram *propagators.*

1.2.4 Expectation values of polynomials

Recall from (1.2) that the quantity that we are trying to compute is not just $\int \mathcal{D}A\, e^{ik\mathcal{L}}$, but it is the expectation value of a certain function $\prod \mathcal{O}_{X_\gamma,R_\gamma}(A)$ of A with respect to the measure $e^{ik\mathcal{L}}\mathcal{D}A$. The functions $\mathcal{O}$ are written explicitly in (1.3) in terms of their Taylor series expansion. Therefore, to understand the integral (1.2) we first have to understand integrals as in (1.7), only with an additional polynomial $P(\vec{x})$ multiplying the integrand. Moreover, after rescaling $\vec{x}$ as in (1.8) and carrying out exactly the same analysis as in (1.9) – (1.10) with an additional $P(\vec{x})$ multiplying each integrand we see that in the mth order term in our revised asymptotic expansion will be given by:

$$\sum_{m=m_1+m_2} \int_{\mathbf{R}^N} d^N x\, e^{i\frac{1}{2}\lambda_{ij}x^i x^j} P_{m_1}(\vec{x})(\lambda_{ijk}x^i x^j x^k)^{m_2},$$

where $P_{m_1}(\vec{x})$ denotes the part of $P(\vec{x})$ which is homogeneous of degree m_1 in $\vec{x}$. Noticing that just as before we ended up having to calculate the expectation value of a polynomial $\left(P_{m_1}(\vec{x})(\lambda_{ijk}x^i x^j x^k)^{m_2}\right)$ with respect to a Gaussian measure, we can now use the same tricks and replace the above integral by a sum of 'revised' Feynman diagrams that are also allowed to have a single exceptional vertex of some order m_1, weighted by the coefficients of $P_{m_1}(\vec{x})$.

1.3 The gauge-fixed Lagrangian

1.3.1 Gauge invariance

Recall that M^3 is an oriented three manifold, G is a Lie group with an invariant integral bilinear form $\mathfrak{tr}$ on its Lie algebra $\mathcal{G}$ and $P \to M^3$ a principal G-bundle on M^3.

The Chern-Simons Lagrangian $cs(A)$ is defined for a connection[4] A by:

$$cs(A) = \frac{1}{4\pi}\int_{M^3} \mathfrak{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

where $\mathfrak{tr}(A_1 \wedge A_2 \wedge A_3) \stackrel{\text{def}}{=} \frac{1}{2}(\mathfrak{tr} A_1 \wedge [A_2, A_3]) = \frac{1}{2}\mathfrak{tr}([A_1, A_2] \wedge A_3)$, and so relative to some choice of coordinates and a trivialization of P,[5]

$$cs(A) = \frac{1}{8\pi}\int_{M^3} \epsilon^{ijk}\mathfrak{tr}(A_i(\partial_j A_k - \partial_k A_j) + \frac{2}{3}A_i[A_j, A_k]).$$

It is invariant under infinitesimal gauge transformations in which $\delta A = -Dc \stackrel{\text{def}}{=} -(dc + [A, c])$:

$$\begin{aligned}
4\pi\delta cs &= -\int_{M^3} \mathfrak{tr}\,((dc + [A,c]) \wedge dA + A \wedge d[A,c] \\
&\qquad\qquad +2(dc + [A,c]) \wedge A \wedge A) \\
&= -\int_{M^3} \mathfrak{tr}([A,c] \wedge dA + A \wedge [dA, c] - A \wedge [A, dc] + 2dc \wedge A \wedge A) \\
&\qquad\qquad -2\int_{M^3} \mathfrak{tr}[A,c] \wedge A \wedge A \\
&= \int_{M^3} \mathfrak{tr}\, c \wedge [A, [A, A]] = 0.
\end{aligned}$$

[4]We will be slightly imprecise and regard A as a $\mathcal{G}$-valued 1-form on M^3.

[5]In the formula below ϵ^{ijk} denotes the totally antisymmetric tensor in three dimensions — $\epsilon^{ijk} = \text{sign}\,(ijk)$ if ijk is a permutation of $\{1,2,3\}$ and $\epsilon^{ijk} = 0$ otherwise.

This implies that $cs(A)$ is invariant under gauge transformations that can be pathwise connected to the identity transformation. As it turns out (see [17]), $cs(A)$ is *not* invariant under general gauge transformations and, in fact, in our normalization it is defined only up to a multiple of 2π. This explains our choice of the normalization — we have chosen precisely that normalization for which the exponential in (1.1) *is* well defined.

The gauge invariance of $cs(A)$ has an unpleasant consequence — the stationary points of are necessarily not isolated, and the quadratic part of $cs(A)$ near a stationary point cannot be non-degenerate. The discussion of Feynman diagrams in the previous chapter depended in an essential way on the invertability of that quadratic part, and therefore cannot be applied here without modification.

1.3.2 The Faddeev-Popov procedure

To resolve the above complication we will once again look at our finite dimensional analogue, assume that the Lagrangian there, $\frac{1}{2}\lambda_{ij}x^ix^j + \lambda_{ijk}x^ix^jx^k$, is invariant under the isometrical non-degenerate action of an l-dimensional Lie group G, and try to evaluate the integral (1.7) without redundant integration over the orbits of G.

We will visit each orbit of G just once by choosing a function $F : \mathbf{R}^n \to \mathbf{R}^l$ that has a unique zero on each G-orbit, and inserting a $\delta^l(F(\vec{x}))$ into the integral. If we want the result to be the same as the full integration and independent of F we need to add a correction term that corresponds to the volume of the G-orbit through $\vec{x}$ and as the action of G is by isometries this term can be calculated locally at a point $\vec{x}$ satisfying $F(\vec{x}) = 0$. It is given by the inverse ratio of the volume element of the Lie-algebra $\mathcal{G}$ of G and its image in $\mathbf{R}^l$ under the action of G composed with F. That is to say — we have to look at[6]:

$$\mathcal{Z} = \int_{\mathbf{R}^N} d^N x \, e^{ik(\frac{1}{2}\lambda_{ij}x^ix^j + \lambda_{ijk}x^ix^jx^k)} \delta^l(F(\vec{x})) \det\left(\frac{\partial F^a}{\partial \mathcal{G}_b}\right)(\vec{x}).$$

($\{\mathcal{G}_b\}_{b=1}^l$ is a set of generators for $\mathcal{G}$)

We will try to find a diagrammatic representation for the asymptotic expansion of $\mathcal{Z}$. The first additional term in the integral is easy — we can just replace it by its Fourier representation:

$$\delta^l(F(\vec{x})) = \int_{\mathbf{R}^l} d^l\phi \, e^{iF^a(\vec{x})\phi_a}$$

and then incorporate $F^a(\vec{x})\phi_a$ as a new term in the Lagrangian. The other new term, $\det\left(\frac{\partial F}{\partial \mathcal{G}}\right)$, can be dealt with in two equivalent ways. The first way is to do the usual rescaling (1.8) and then to expand $\det\left(\frac{\partial F}{\partial \mathcal{G}}\right)$ in powers of $\frac{1}{\sqrt{k}}$ by first separating $\det\left(\frac{\partial F}{\partial \mathcal{G}}\right)$ to a constant part J_0 and a part $J_1(\vec{x})$ which is a series in $\frac{1}{\sqrt{k}}$, and then

[6]This expression for $\mathcal{Z}$ was first derived by Faddeev and Popov in [20].

using

$$\det\left(J_0 + \frac{1}{\sqrt{k}} J_1(\vec{x})\right) = \det(J_0) \sum_m \left(\frac{1}{\sqrt{k}}\right)^m tr(\bigwedge^m J_0^{-1})(\bigwedge^m J_1(\vec{x})). \tag{1.12}$$

($\bigwedge^m J$ is the mth exterior power of the matrix J). Notice that J_0 is just a constant matrix, while $J_1(\vec{x})$ depends on $\vec{x}$. It will now be possible to regard (1.12) as a polynomial in $\vec{x}$ and get a Feynman diagram expansion. It is an exercise in elementary algebra to show that the polynomial (1.12) can itself be incorporated into the the Feynman diagrams by introducing a new type of propagator denoted by *directed* dotted lines that corresponds to J_0^{-1} and a collection of new types of vertices each connecting two dotted propagators with some dashed propagators — depending on the exact form of $J_1(\vec{x})$. (There will also be some alteration to the combinatorial rule of determining the numerical factor multiplying each diagram).

The other way of dealing with $\det\left(\frac{\partial F}{\partial \mathcal{G}}\right)$ is the one commonly used in the physics literature and the one that we will be using here. It involves the idea of anti-commutative integration. Non-commutative integration is treated in many places (see e.g. [9, 36, 21, 29]), and I will not explain it here in detail. Very briefly, 'anti-commuting' variables (called 'ghosts') $\{\bar{c}_a\}_{a=1}^l$ and $\{c^b\}_{b=1}^l$ are introduced together with a reasonable set of rules of integration with respect to them, and it is shown that for any matrix $J^a{}_b$

$$\int d^l\bar{c}d^lce^{\bar{c}_aJ^a{}_bc^b} \propto \det(J). \tag{1.13}$$

(This is analogous and complementary to standard Gaussian integration — in which the resulting determinant is in the denominator).

Using this, $\mathcal{Z}$ can finally be written as

$$\mathcal{Z} \propto \int_{\mathbf{R}^N} d^Nx \int_{\mathbf{R}^l} d^l\phi \int d^l\bar{c}\, d^lc\, e^{i(k(\frac{1}{2}\lambda_{ij}x^ix^j + \lambda_{ijk}x^ix^jx^k) + F^a(\vec{x})\phi_a) + \bar{c}_a(\frac{\partial F^a}{\partial \mathcal{G}_b})c^b} = \int e^{i\mathcal{L}_{tot}}.$$

Now we can use almost the same procedure as in (1.9) – (1.11) to get a diagrammatic expansion for the asymptotic behavior of $\mathcal{Z}$. Again it turns out that this involves introducing a new propagator and some new vertices.

As we will see below for the case of interest for us — the Chern-Simons Lagrangian — we will be able to choose F in a way so that the quadratic part of the supplemented Lagrangian will indeed be invertible.

1.3.3 Gauge-fixing for the Chern-Simons action

Let A_0 be an arbitrary stationary point for cs, i.e.: $\frac{\delta cs}{\delta A}(A_0) = 0$, which means $F^{A_0} = dA_0 + \frac{1}{2}[A_0, A_0] = dA_0 + A_0 \wedge A_0 = 0$, let D denote the exterior derivative d twisted by A_0, and for A an $ad(P)$-valued 1-form on M^3 define $\mathcal{L}(A) = cs(A_0 + A)$:

$$\mathcal{L}(A) = cs(A_0 + A) = cs(A_0) + \frac{1}{4\pi}\int_{M^3} \mathrm{tr}(A \wedge DA + \frac{2}{3}A \wedge A \wedge A).$$

Choose a trivialization of P, local coordinates $\{x^i\}$ and a metric g_{ij} on M^3 with $g \stackrel{\text{def}}{=} \det(g_{ij})$, and get

$$(DA)_{ij} = \partial_i A_j - \partial_j A_i + [A_{0i}, A_j],$$

and

$$D^i \stackrel{\text{def}}{=} \sqrt{g} g^{ij} D_j = \sqrt{g} g^{ij} \partial_j + \sqrt{g} g^{ij} [A_{0j}, \cdot].$$

Pick the gauge condition $\frac{k}{2\pi} D_i A^i = 0$, and get using the Faddeev-Popov procedure as described in the previous subsection:

$$\begin{aligned}
\mathcal{L}_{tot}(A, \phi, c, \bar{c}) &= k\mathcal{L} + \mathcal{L}_{gauge-fixing} + \mathcal{L}_{ghosts} \\
&= kcs(A_0) + \frac{k}{4\pi} \int_{M^3} \mathfrak{tr}(A \wedge DA + \frac{2}{3} A \wedge A \wedge A) \\
&\quad + \frac{k}{2\pi} \int_{M^3} \mathfrak{tr} \left(\phi D_i A^i - i\bar{c} D_i (D^i + \text{ad } A^i) c \right)
\end{aligned} \tag{1.14}$$

ϕ, c, and $\bar{c}$ are Lie-algebra valued fields — $\phi = \phi^a \mathcal{G}_a$, $c = c^a \mathcal{G}_a$, and $\bar{c} = \bar{c}^a \mathcal{G}_a$ for a set of generators $\{\mathcal{G}_a\}$ of $\mathcal{G}$.

Chapter 2

The Feynman rules

In this chapter we will write the Feynman rules for the Chern-Simons theory, defined by the total Lagrangian (1.14). Looking at (1.14) we see that the quadratic part of our total Lagrangian decouples to a sum of two quadratic forms, one involving A and ϕ, and one involving $\bar{c}$ and c. Therefore, in the diagrammatic expansion of $\int e^{i\mathcal{L}_{tot}}$ there will be *two* types of propagators — a dashed line $(----)$ representing the inverse of the $A\phi$ quadratic form, and a directed solid line $(\longrightarrow)$ representing the inverse of the $\bar{c}c$ quadratic form. One can also see that the cubic part of $\mathcal{L}_{tot}$ is the sum of two terms. The first of these two terms is $\frac{2}{3}A \wedge A \wedge A$ and it corresponds to an order 3 vertex connecting three dashed lines. The second is $\bar{c}D_i[A^i, c]$ and it corresponds to an order 3 vertex connecting an incoming directed solid line, an outgoing directed solid line, and a dashed line. Also, recall that we are not just computing $\int e^{i\mathcal{L}_{tot}}$, but something slightly more complicated — $\int \prod \mathcal{O} e^{i\mathcal{L}_{tot}}$. Looking at equation (1.3), we see that the inclusion of the $\mathcal{O}$'s amounts to adding a vertex of a third type in which a dashed line ends on an ellipse that represents a component of the knot.

Other then what was said above, I will skip the precise derivation of the Feynman rules, and just describe the end result in the next few pages. For simplicity we will restrict our attention to the case of a single (directed) knot $\mathcal{X} = \{X\}$. There is no difficulty to restrict the rules given below to the case were there is no knot and we are trying to compute a 3-manifold invariant, or to enhance these rules to the case of a many-component link. X will be given by a parametrization $X(s) : S^1 \to M^3$, where S^1 is the oriented unit circle.

2.1 The diagrams

Pick an integer m, *the order, the number of loops.* To obtain the m'th invariant $\mathcal{W}_m(X)$, first write all inequivalent connected[1] *Feynman diagrams* of order m. A

[1] Restricting our attention to connected diagrams corresponds to computing the asymptotics of $\mathcal{W}(M^3, X, k)/\mathcal{W}(M^3, k)$ instead of that of $\mathcal{W}(M^3, X, k)$.

Feynman diagram of order m is a diagram made of a single[2] directed ellipse (called *a Wilson loop*) representing the knot $\mathcal{X}$, a total of $2m$ cubic vertices of three different types — *type* X^2A, *type* $\bar{c}Ac$, and *type* A^3, and lines (called *propagators*) connecting those vertices. There are two types of propagators. The *gauge propagators* denoted by dashed lines $- - - -$, and the *ghost propagators* denoted by directed solid lines $\longrightarrow$. The three types of vertices differ by the types of propagators they are allowed to connect. In a type X^2A vertex a gauge propagator meets the Wilson loop representing the knot. A type $\bar{c}Ac$ connects a gauge propagator with one incoming and one outgoing ghost propagators. Finally, in a type A^3 vertex three gauge propagators meet. Figure 2.1 is an example for such a diagram. When looking at that figure, remember that our diagrams are not allowed to have higher than cubic vertices. It is therefore implicitly understood that when four or more lines meet at the same point, that point is not a vertex and those lines pass each other without "interaction".

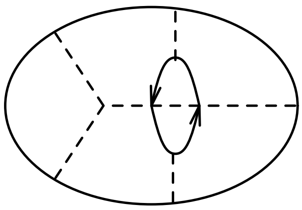


Figure 2.1. An example for a Feynman diagram of order 4, having 5 type X^2A vertices, 2 type $\bar{c}Ac$ vertices, one type A^3 vertex, 5 gauge propagators, and 2 ghost propagators.

Two diagrams are called *equivalent* if one can set a bijective type-preserving correspondence between their vertices, in a way that corresponding vertices are connected by the same type, same orientation, and the same number of propagators and Wilson loop segments.

For example, if $m = 2$, the five[3] diagrams that we write in this stage are illustrated in figure 2.2.

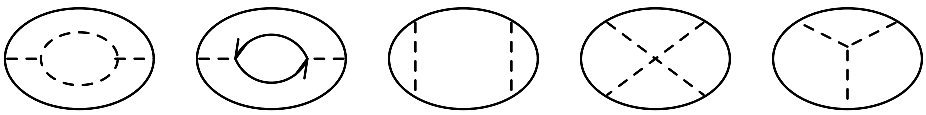


Figure 2.2. The five diagrams of order 2.

[2]Of course, if we were dealing with a link with Γ components we would have had Γ Wilson loops.

[3]Actually, few more such diagrams can be written — but the ones that are not shown in the figure are all *singly connected* — namely, they can be broken apart into two components by the removal of a single arc. It is easy to see that such diagrams have a vanishing Lie-algebraic coefficient if the connection A_0 of the section 1.3.3 is the zero connection on a trivial bundle. We will ignore these diagrams below.

2.2 The procedure

Our invariant $\mathcal{W}_m(X)$ will be a sum of finite dimensional integrals, one corresponding to each Feynman diagram. Let us concentrate in a single diagram D, and see how to write the finite dimensional integral corresponding to it. This will be done in several steps:

1. Mark the parts D as follows. Mark every end of every gauge propagator with a lowercase letter from i, j, ... (thought to represent a spatial index — an integer in $\{1, 2, 3\}$). Mark every type $\bar{c}Ac$ or type A^3 vertex by a letter from x, y, ... (thought to represent a point in M^3). Add a lowercase letter from a, b, ... (thought to represented a basis element of $\mathcal{G}$) to every end of every propagator. Finally pick a base point on the Wilson loop and follow the loop according to its orientation marking the X^2A vertices encountered along the way by s_1, s_2, ... (representing points in the parameter space S^1 of X) and marking the segments of the Wilson loop cut by these vertices by lowercase greek letters such as α, β, ... (thought to represented a basis element of the representation R). For an example, see figure 2.3.

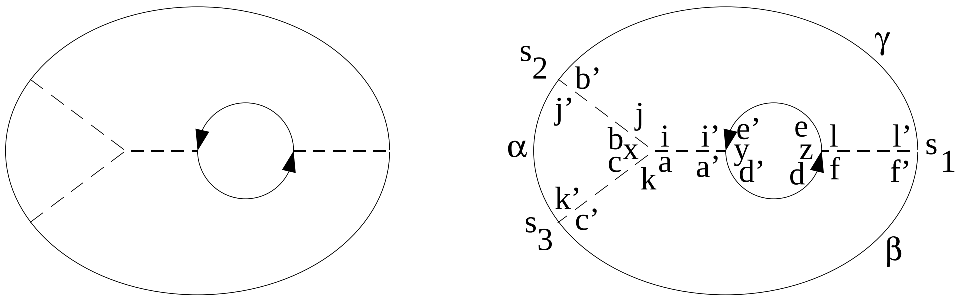


Figure 2.3. An unmarked diagram and its marking.

2. If D is a marked diagram, construct an algebraic expression $\mathcal{E}(D)$ by taking a product of terms, each corresponding to a part of the diagram D as follows:

   (a) For each gauge propagator in D, marked, say, as $\underset{x}{\overset{i,a}{-}} - - \underset{y}{\overset{i',a'}{-}}$ take the term

   $$V_{ii'}^{aa'}(x, y). \tag{2.1}$$

   $V_{IJ}^{ab}(x, y)$ is defined to be the inverse of the bosonic free part of the Lagrangian $\mathcal{L}$. The symbols "I" and "J" are either numbers i, j in the range $1 - 3$, or the symbol ϕ, and with this understood V is defined by the relations: (the differentiations[4] are all with respect to x.)

   $$t_{ab} D_i \sqrt{g} g^{ij} V_{j\phi}^{bc}(x, y) \;=\; 2\pi i \delta_a^c \delta(x, y),$$

[4] Remember that D is the covariant derivative with respect to the connection A_0.

$$
\begin{aligned}
t_{ab} D_i \sqrt{g} g^{ij} V^{bc}_{jk}(x,y) &= 0, \\
t_{ab} \left(\epsilon^{ijk} D_j V^{bc}_{kl}(x,y) + D^i V^{bc}_{\phi l}(x,y) \right) &= 2\pi i \delta^c_a \delta^i_l \delta(x,y), \\
t_{ab} \left(\epsilon^{ijk} D_j V^{bc}_{k\phi}(x,y) + D^i V^{bc}_{\phi\phi}(x,y) \right) &= 0.
\end{aligned}
$$

If one (or both) of the ends of a certain gauge propagator is an $X^2 A$ vertex, marked by a point s in the parameter space of X, for the purposes of this construction simply replace it by the point $X(s) \in M^3$. For example,

$$
\frac{l,f}{z} - - \frac{l',f'}{s_1} \longrightarrow V^{ff'}_{ll'}(z, X(s_1)).
$$

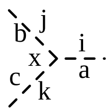
(b) For each ghost propagator in D, marked, say, as $\frac{d'}{y}$———$\frac{d}{z}$ take the term

$$
G^{d'd}(y,z).
$$

G is defined to be the inverse of the ghost free part of the Lagrangian $\mathcal{L}$ — that is to say, it is defined by the relation: (the differentiations are all with respect to x.)

$$
t_{ab} D_i D^i G^{bc} = -2\pi \delta^c_a \delta(x,y).
$$

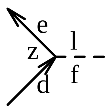
(c) For each marked A^3 vertex in D use the rule



$$
\longrightarrow \frac{i}{2\pi} \int_{M^3} dx\, t_{abc} \epsilon^{ijk}. \tag{2.2}
$$

The symbol t_{abc} essentially represents the structure constants of $\mathcal{G}$ — to define t_{abc} we pick a basis $\{\mathcal{G}_a\}$ of $\mathcal{G}$, compute the structure constants $[\mathcal{G}_a, \mathcal{G}_b] = f^c_{ab}\mathcal{G}_c$ and use the bilinear form $\mathfrak{tr}$ to 'lower' the index c: $t_{abc} = f^d_{ab} t_{dc}$ where $t_{ab} = \mathfrak{tr}(\mathcal{G}_a \mathcal{G}_b)$.

(d) For each $\bar{c}Ac$ vertex in D use the rule

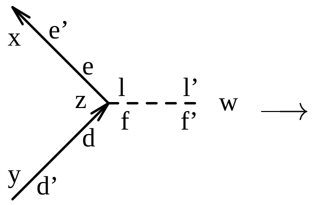


$$
\longrightarrow \frac{1}{2\pi} \int_{M^3} dz\, t_{dfe} D^l_z. \tag{2.3}
$$

Here D^l_z denotes differentiation with respect to z^l,

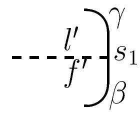
$$
D^l_z = \sqrt{g} g^{lm} \frac{D}{Dz^m}
$$

acting only on the z-dependence of the term coming from the ghost propagator leaving the vertex. For a better understanding, let us look at this term together with the terms corresponding to the propagators surrounding our vertex:



$$\longrightarrow \quad \frac{1}{2\pi}\int_{M^3} dz\, t_{dfe}\left(\sqrt{g}g^{lm}\frac{D}{Dz^m}G^{ee'}(z,x)\right)$$
$$\cdot G^{d'd}(y,z)V^{ll'}_{ff'}(z,w)$$

(e) For each marked X^2A vertex in D use the rule



$$\longrightarrow -R^{\beta}_{f'\gamma}\dot{X}^{l'}(s_1). \tag{2.4}$$

Here $R^{\alpha}_{a\beta}$ is simply the representation R expressed in terms of matrices — if $\{r^{\alpha}\}_{\alpha=1}^{\dim R}$ is a basis of R, then $R(\mathcal{G}_a)r^{\alpha} = R^{\alpha}_{a\beta}r^{\beta}$.

(f) Notice that by the restrictions we have on the types of allowed vertices in D, the ghost propagators must form a set of disjoint closed loops. The last term in $\mathcal{E}(D)$ will be

$$(-1)^F, \tag{2.5}$$

where F is the number of such loops.

3. Now integrate the variables s_1, s_2, ... over S^1 *preserving their cyclic order.*

4. Divide the resulting integral by a combinatorial factor, $S(D)$. For a diagram D, $S(D)$ is the total number of symmetries of D. A symmetry of a digram D is a bijective self-map on the set of vertices and arcs in D, which sends a vertex to a vertex of the same type, a propagator to a propagator of the same type, a Wilson loop segment to a Wilson loop segment, and preserves beginning and end points — the image of the beginning and end points of an arc have to be the beginning and end points of the image of that arc. For example, the weights $S(D)$ of the five diagrams in figure 2.2 are 4, 2, 2, 4, and 3 respectively, while that of the diagram in figure 2.3 is 1.

Example The complete expression corresponding to the diagram in figure 2.3 is

$$\frac{i}{8\pi^3}\int_{s_1<s_2<s_3} ds_{1-3}\int d^3x d^3y d^3z\; t_{a'd'e'}t_{dfe}t_{abc}\epsilon^{ijk}R^{\beta}_{f'\gamma}R^{\gamma}_{b'\alpha}R^{\alpha}_{c'\beta}$$
$$\cdot\dot{X}^{l'}(s_1)\dot{X}^{j'}(s_2)\dot{X}^{k'}(s_3)\left(D^{i'}_yG^{d'd}(y,z)\right)\left(D^l_zG^{ee'}(z,y)\right)$$
$$\cdot V^{aa'}_{ii'}(x,y)V^{ff'}_{ll'}(z,X(s_1))V^{bb'}_{jj'}(x,X(s_2))V^{cc'}_{kk'}(x,X(s_3))$$

(In this integral the domain definition $s_1 < s_2 < s_3$ should be read as 'the set of all $s_{1,2,3} \in S^1$ for which s_2 is between s_1 and s_3 in the chosen orientation of S^1', and not as a linear ordering relation).

Chapter 3

The one-loop contribution

3.1 When $M^3 = \mathbf{R}^3$

Having developed a general technique in the previous chapters, let us now try to apply it in few particular cases, and let us start from the simplest case — the contribution of order $1/k$ to $\mathcal{W}(\text{flat } \mathbf{R}^3, \mathcal{X})$ where $\mathcal{X}$ is a one- or two-component link in $\mathbf{R}^3$. There is just one flat connection on $\mathbf{R}^3$ — the trivial one — and we take it to be the background connection A_0. In this simple case the ghosts and the interaction term $A \wedge A \wedge A$ don't yet come into play, and of the infinitely many terms in the expansion of $\mathcal{P}$exp only terms up to the second order term will be relevant. That is to say, we just need to understand

$$\begin{aligned}\mathcal{W}' \; = \; & \int_{\mathcal{A}} \mathcal{D}A\mathcal{D}\phi \; e^{\frac{ik}{4\pi}\int_{\mathbf{R}^3} \mathrm{tr}(\epsilon^{ijk}A_i\partial_j A_k + 2\phi\partial^i A_i)} \\ & \prod_{\gamma=1}^{2} \Big(\dim R_\gamma + \int ds \dot{X}^i_\gamma(s) A^a_i(X_g(s)) R^\alpha_{\gamma a\alpha} \\ & \quad + \int_{s_1<s_2} ds_{1,2} \dot{X}^{i_1}_g(s_1)\dot{X}^{i_2}_\gamma(s_2) A^{a_1}_{i_1}(X_\gamma(s_1)) A^{a_2}_{i_2}(X_\gamma(s_1)) R^{\alpha_1}_{\gamma a_1\alpha_2} R^{\alpha_2}_{\gamma a_2\alpha_1}\Big)\end{aligned}$$

This is just a simple Gaussian integral. We can regard ϕ as a (Lie algebra valued) three-form on $\mathbf{R}^3$, A as a one-form, and write the quadratic form in our Gaussian integral as

$$\frac{1}{2}\frac{i}{2\pi}\int_{\mathbf{R}^3} \mathrm{tr}\left(\epsilon^{ijk}A_i\partial_j A_k + 2\phi\partial^i A_i\right) = \frac{1}{2}\frac{i}{2\pi}\left\langle \begin{pmatrix} A \\ \phi \end{pmatrix}, L_- \begin{pmatrix} A \\ \phi \end{pmatrix} \right\rangle$$

for $L_- \stackrel{\text{def}}{=} (d\star + \star d)J$, where $JA \stackrel{\text{def}}{=} A$ and $J\phi \stackrel{\text{def}}{=} -\phi$. Clearly $(L_-)^2 = \Delta$ and therefore V, which is essentialy the inverse of L_-, is given by $V = 2\pi i L_- \circ G_\Delta$ where G_Δ is the Green's function of the (vector + scalar) Laplacian Δ. In the Euclidean case this Green's function G_Δ is given by

$$G^{ab}_\Delta(x,y) = \frac{t^{ab}}{4\pi|x-y|} \qquad (t^{ab} \text{ is the inverse of } t_{ab} \stackrel{\text{def}}{=} \mathrm{tr}(\mathcal{G}_a\mathcal{G}_b))$$

for both the scalar and the vector cases, and so the A part of our propagator is given by

$$\underset{a,i}{\overset{x}{-}} - - \underset{b,j}{\overset{y}{-}} V_{ij}^{ab}(x,y) = \langle A_i^a(x) A_j^b(y) \rangle = 2\pi i \epsilon_{ikj} \partial_x^k \frac{t^{ab}}{4\pi|x-y|} = \epsilon_{ijk} t^{ab} \frac{i(x-y)^k}{2|x-y|^3}.$$

The terms of order $1/k$ are given by the diagrams in figure 3.1.

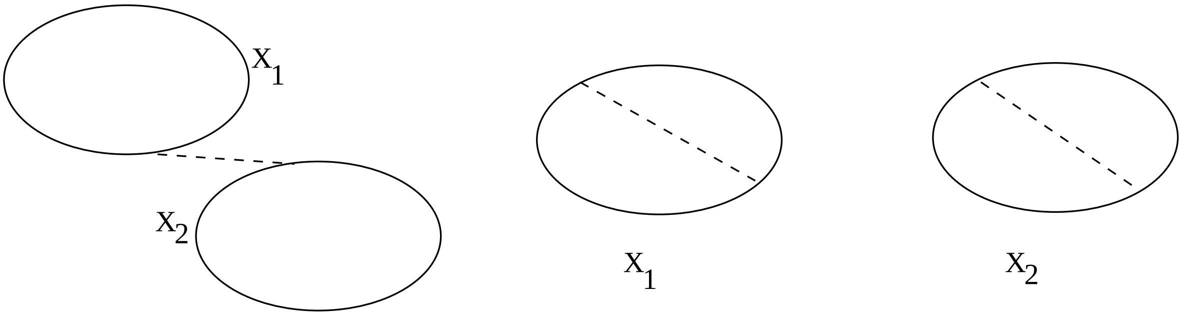


Figure 3.1. First order diagrams

3.2 The linking number of two knots

Let us first consider the left most diagram. Ignoring the constant numerical coefficient that the representations $R_{1,2}$ contribute it corresponds to the integral

$$£(X_1, X_2) = \int ds_1 ds_2 V_{ij}(X_1(s_1), X_2(s_2)) \dot{X}_1^i \dot{X}_2^j \tag{3.1}$$

which is the well known Gauss integral representation for $2\pi i$ times the linking number of two knots [38]. For the sake of completeness, and also as a preparation for the next chapter where we will use similar but more complicated considerations to deal with the two loop contribution, we will review here the proof of the invariance of (3.1) under isotopies and show that indeed it coincides with the linking number.

It is possible to view $V_{ij}(x,y)$ is as a $(1,1)$-form[1] on $\mathbf{R}^3 \times \mathbf{R}^3$ where $(x,y) \in \mathbf{R}^3 \times \mathbf{R}^3$, i is the one form index for the variable x, and j is the one form index for the variable y. Viewed this way, (3.1) is just that form V evaluated on the cycle X_1 relative to its left variable and on the cycle X_2 relative to its right variable:

$$£ = \langle X_1 | V | X_2 \rangle$$

The key property required for the invariance proof is that there exists a $(2,0)$-form F for which

$$d^L V = d^R F \tag{3.2}$$

[1] An (m,n)-form on $M \times N$ where M and N are smooth manifolds is a section of $\pi_M^* TM \otimes \pi_N^* TN$ where $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are the projections. Clearly, one can define operators $d^L : \{(m,n)\text{-forms}\} \to \{(m+1,n)\text{-forms}\}$, $d^R : \{(m,n)\text{-forms}\} \to \{(m,n+1)\text{-forms}\}$, etc. in analogy with the standard definitions of de-Rham theory.

away from the diagonal, where d^L is the exterior derivative with respect to the left variable and d^R is the exterior derivative with respect to the right variable. Assuming such an F, under an infinitesimal deformation of X_1 we will have (using Stoke's theorem twice)

$$\delta £ = \delta\langle X_1|V|X_2\rangle = \langle{}^{\text{The surface } S \text{ spanned by the}}_{\text{infinitesimal deformation of } X_1}|d^L V|X_2\rangle = \langle S|d^R F|X_2\rangle = 0. \quad (3.3)$$

As for the existence of F, notice that by our derivation of V, $V = 2\pi i \star d \circ G_v$ where G_v is the vector part of G_Δ, and therefore $\star^L d^L V = 2\pi i \star d \star d \circ G_v$. By the commutativity of $\star d$ and G_v one gets $\star^L d^L V = 2\pi i G_v \circ \star d \star d$. Remembering that G_v is given by an integral kernel, one can integrate by parts $G_v \circ \star d \star d$ to get $\star^L d^L V = 2\pi i \star^R d^R \star^R d^R G_v = 2\pi i(\Delta^R + d^R \star^R d^R \star^R)G_v = 2\pi i I + 2\pi i d^R \star^R d^R \star^R G_v$. Multiplying from the left by $\star^L$ we obtain:

$$d^L V = d^R 2\pi i \star^L \star^R d^R \star^R G_v + 2\pi i \star^L I \overset{\text{def}}{=} d^R F + 2\pi i \star^L I.$$

The formula we just got for F can be expanded to give

$$F_{ij,-}(x,y) = \epsilon_{ijk}\frac{i(x-y)^k}{2|x-y|^3},$$

and this can be used to verify (3.2) directly. Don't let yourself be mislead by the apparent equivalence of the formulae for V and for F! The indices are arranged a little differently and verifying (3.2) is a little more than just playing around with these indices — some differentiations do have to be carried out and the verification *is* essentially the same calculation as the derivation in this paragraph.

Having shown that £ is indeed an isotopy invariant we can now use it to show that it coincides with $2\pi i$ times the linking number. Deform the knot so that it will be almost planar with only 'perpendicular crossings'. Now flip one of those crossings us shown in figure 3.2. Clearly, when comparing the contribution to £ from before

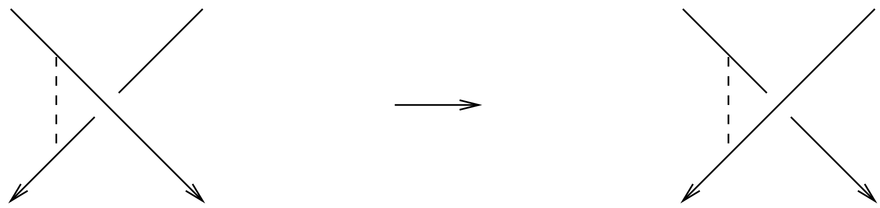


Figure 3.2. Flipping a crossing

and from after the flip we can integrate the propagator with its endpoints only nearby the crossing. If the crossed arcs are ϵ apart,

$$£(\text{after}) - £(\text{before}) = i\int ds_{1,2}\frac{\epsilon}{(\epsilon^2+s_1^2+s_2^2)^{3/2}} = 2\pi i. \quad (3.4)$$

This is exactly the same relation is satisfied by $2\pi i$ times the linking number, and together with £(unlinked circles) $= 0$ (3.4) proves that £ is indeed $2\pi i$ times the linking number. To see that indeed £(unlinked circles) $= 0$, use the already proven isotopy invariance to make sure that the two circles are very small relative to the separation between them and then the integral defining £ will tend to zero.

3.3 The self-linking of a single knot

The situation with the other diagrams in figure 3.1 is a bit more complicated. Let $£_s(X_1)$ be the 'self-linking' of X_1:

$$£_s(X_1) \stackrel{\text{def}}{=} \frac{1}{2}\langle X_1|V|X_1\rangle = \frac{1}{2}\int ds_1 ds_2 V_{ij}(X_1(s_1), X_1(s_2))\dot{X}_1^i(s_1)\dot{X}_1^j(s_2). \tag{3.5}$$

(We have suppressed here the Lie-algebra coefficient which for R being the defining representation of $G = SU(N)$ in $\mathbf{C}^N$ and for $\mathfrak{tr}$ being the usual matrix trace can be seen to equal $N^2 - 1$. For more details see chapter 9).

For three vectors A, B, C it will be convenient to denote $\epsilon_{ijk}A^iB^jC^k$, the volume of the parallelepiped spanned by $\vec{0}, A, B, C$ by $\det(A|B|C)$. Using this notation

$$£_s(X) = \frac{i}{4}\int ds_1 ds_2 \frac{\det\left(X(s_1) - X(s_2)|\,\dot{X}(s_1)\,\middle|\dot{X}(s_2)\right)}{|X(s_1) - X(s_2)|^3}. \tag{3.6}$$

This integral appears at first sight to be divergent because of the cubic term in the denominator. Nevertheless when s_1 and s_2 are close, say ϵ apart, $X(s_1) - X(s_2) \sim \epsilon$ and the three vectors $X(s_1) - X(s_2)$, $\dot{X}(s_1)$, and $\dot{X}(s_2)$ are within a cone of opening $\sim \epsilon$. Therefore the volume of the parallelepiped spanned by these three vectors is $\sim \epsilon^3$ which is enough to suppress the singularity of the denominator. Unluckily, the argument in (3.3) doesn't go through — the key relation (3.2) holds only away from the diagonal, and in (3.5) our integration domain does intersect the diagonal.

This point has already been treated by Căalugăreanu [13, 14] (see also Pohl [34]) and later from a physical viewpoint by Polyakov [35] (see also Tze [40]). They found that indeed (3.5) is not an invariant, but yet it can be renormalized by the addition of a local term (essentially the total torsion of X) to give an invariant. It turns out that to properly define the torsion everywhere X needs to be 'framed', and therefore $£_s$ will just be an invariant of *framed* knots. We will arrive at the same results using a somewhat different regularization which makes the current calculation a bit less transparent but has a more straightforward generalization for the two-loop case to be treated in the next chapters. Let us define $£_\epsilon$ by the integral (3.6) that defines $£_s$, only with the integration domain restricted to

$$\Delta_\epsilon \stackrel{\text{def}}{=} [|s_1 - s_2| > \epsilon].$$

Assume that X undergoes an infinitesimal deformation $X \to X + \delta X \stackrel{\text{def}}{=} X + \omega$. As in the invariance proof for the case of a link, (3.3), Stoke's theorem was used twice it will fail twice for this new case and $\delta\pounds_\epsilon$ will pick up four non-zero contributions — one from each boundary term in each of the usages of Stoke's theorem. Denoting the evaluation of differential forms on Δ_ϵ by $\langle\ |\ |\ \rangle_\Delta$ and on its two boundaries $[s_1 - s_2 = \pm\epsilon]$ by $\langle\ |\ |\ \rangle_\pm$ we will get: (S again is the surface spanned by the infinitesimal deformation of X)

$$\begin{aligned}\delta\pounds_\epsilon &= \frac{1}{2}\delta\langle X|V|X\rangle_\Delta \\ &= \langle S|d^L V|X\rangle_\Delta + \langle\omega|V|X\rangle_+ - \langle\omega|V|X\rangle_- \\ &= \langle S|d^R F|X\rangle_\Delta + \langle\omega|V|X\rangle_+ - \langle\omega|V|X\rangle_- \\ &= \langle S|F|-\rangle_+ - \langle S|F|-\rangle_- + \langle\omega|V|X\rangle_+ - \langle\omega|V|X\rangle_- .\end{aligned} \tag{3.7}$$

We will try to understand the $\epsilon \to 0$ limit of $\delta\pounds_\epsilon$ by expanding (3.7) in powers of ϵ. For s a variable in S^1 let $X = X(s)$, $\dot{X} = \dot{X}(s)$, $\omega = \omega(s), \ldots$,

$$\begin{aligned} X_{\pm\epsilon} = X(s\pm\epsilon) &\sim X \pm \epsilon\dot{X} + \frac{\epsilon^2}{2}\ddot{X} \pm \frac{\epsilon^3}{6}\dddot{X} \\ \dot{X}_{\pm\epsilon} = \dot{X}(s\pm\epsilon) &\sim \dot{X} \pm \epsilon\ddot{X} + \frac{\epsilon^2}{2}\dddot{X}\end{aligned}$$

Using these notations, with the dummy integration variable s picked to be at the point where ω is evaluated,

$$\begin{aligned}\langle\omega|V|X\rangle_\pm &= \frac{i}{2}\int ds \frac{\det\left(\dot{X}_{\pm\epsilon}|\omega|X_{\pm\epsilon} - X\right)}{|X - X_{\pm\epsilon}|^3} \\ \sim \frac{i}{2}\int ds &\frac{\det\left(\dot{X} \pm \epsilon\ddot{X} + \frac{\epsilon^2}{2}\dddot{X}|\omega| \pm \epsilon\dot{X} + \frac{\epsilon^2}{2}\ddot{X} \pm \frac{\epsilon^3}{6}\dddot{X}\right)}{|X - X_{\pm\epsilon}|^3} \\ &\sim \frac{i}{2}\int ds \frac{\epsilon^2 \det\left(\frac{1}{2}\ddot{X} \pm \frac{\epsilon}{3}\dddot{X}|\omega|\dot{X} \pm \frac{\epsilon}{2}\ddot{X}\right)}{|\epsilon|^{-3}|\dot{X}|^{-3}\left(1 \mp \epsilon\frac{3\dot{X}\cdot\ddot{X}}{2|\dot{X}|^2}\right)}.\end{aligned}$$

Therefore (notice that the terms of order $\frac{1}{\epsilon}$ cancel!)

$$\langle\omega|V|X\rangle_+ - \langle\omega|V|X\rangle_- \sim \frac{i}{2}\int \frac{ds}{|\dot{X}|^3}\left(-\frac{3\dot{X}\cdot\ddot{X}}{2|\dot{X}|^2}\det(\ddot{X}|\omega|\dot{X}) + \frac{2}{3}\det(\dddot{X}|\omega|\dot{X})\right).$$

Similarilly

$$\begin{aligned}\langle S|F|-\rangle_\pm &= -\frac{i}{2}\int ds \frac{\det(\dot{X}|\omega|X_{\pm\epsilon} - X)}{|X_{\pm\epsilon} - X|^3} \\ \sim -\frac{i}{2}\int ds &\frac{|\dot{X}|^3}{|\epsilon|}\left(1 \pm \epsilon\frac{3\dot{X}\cdot\ddot{X}}{2|\dot{X}|^2}\right)\det\left(\dot{X}\middle|\omega\middle|\frac{1}{2}\ddot{X} \mp \frac{\epsilon}{6}\dddot{X}\right)\end{aligned}$$

and therefore (notice that again there is no term of order $\frac{1}{\epsilon}$)

$$\langle S|F|-\rangle_+ - \langle S|F|-\rangle_- \sim \frac{i}{2}\int \frac{ds}{|\dot{X}|^3}\left(-\frac{3\dot{X}\cdot\ddot{X}}{2|\dot{X}|^2}\det(\ddot{X}|\omega|\dot{X}) + \frac{1}{3}\det(\dddot{X}|\omega|\dot{X})\right).$$

This finally gives that the $\epsilon \to 0$ limit of $\delta \pounds_\epsilon$ is[2]

$$\delta \pounds_s = \frac{i}{2} \int \frac{ds}{|\dot{X}|^3} \left(-3 \frac{\dot{X} \cdot \ddot{X}}{|\dot{X}|^2} \det(\ddot{X}|\omega|\dot{X}) + \det(\dddot{X}|\omega|\dot{X}) \right) \tag{3.8}$$

and we can see that indeed $\delta \pounds_s \neq 0$ and $\pounds_s$ is *not* a knot invariant.

3.4 The appearance of framings

Yet, some further investigation of $\delta \pounds_s$ shows that this can be corrected quite easily. Define τ to be $i/2$ times the total torsion of the curve X — that is to say $i/2$ times the integral with respect to arc length of the local torsion $\tau(s)$ (see [18, pp. 22]) of the curve, given by the standard formula

$$\tau(s) = -\frac{\det(\dot{X}(s)|\ddot{X}(s)|\dddot{X}(s))}{|\dot{X}(s) \times \ddot{X}(s)|^2} \tag{3.9}$$

whenever the denominator is non-zero. As I will comment below, under $X \to X + \omega$ one can show that $\delta \pounds_s$ and $-\delta\tau$ are given by exactly the same formula (3.8) so if one defines

$$\pounds_r = \pounds_s + \tau$$

then $\pounds_r$ *is* invariant under isotopies, so long as the denominator in (3.9) remains non-zero.

What if that denominator *is* equal to zero? On the normal bundle of X there is a canonically defined connection defined by the projection back to the normal bundle of the usual differentiation along the knot of vector functions normal to it. $i/2$ times the total holonomy of that connection along the knot is some imaginary number, well defined up to a multiple of πi which depends on a choice of a trivialization for the normal bundle, and whenever τ is defined, it will be shown below to coincide with that number. Hence $\pounds_r$ is an invariant of *framed* knots — a framing is just a trivialization of the normal bundle which can be used to render τ and therefore $\pounds_s$ well-defined. This necessity of framing the knot X agrees with the results of Witten [42], but makes $\pounds_r$ quite useless for an unframed knot — it is a multiple of πi which is well-defined only up to a multiple of πi. For a framed knot it can be shown along the same lines as in (3.4) to be πi times the self-linking of a framed knot — πi times the linking number of that knot with its framing.

To complete the discussion we need to demonstrate the two differential geometric assertions made above. Very briefly, if $n(s)$ is *any* vector not tangent to the knot X then the holonomy discussed above can be calculated by measuring how much the

[2] It is not hard to verify that the operations of taking the variation under $X \to X + \omega$ and of taking the $\epsilon \to 0$ limit commute. A harder check of the same kind is described at the end of section 4.3.3.

projection of n to the normal bundle fails to be parallel. It is an elementary exercise to then find that relative to the framing given by n,

$$\tau = \frac{-i}{2} \int ds |\dot{X}| \det \left(\frac{\dot{X}}{|\dot{X}|^2} \middle| n \middle| \frac{|\dot{X}|^2 \dot{n} - (\dot{X} \cdot n) \ddot{X}}{|\dot{X} \times n|^2} \right). \tag{3.10}$$

Setting $n = \ddot{X}$ it is easy to see that (3.10) coincides with (3.9) and choosing n to be a constant vector that is not parallel to $\dot{X}(s)$ for any value of s simplifies it the most. One can then vary (3.10) under $X \to X + \omega$ and integrate by parts until all the derivatives of ω disappear. One is left with a huge and unfriendly expression that with a tremendous amount of labor and juggling with vector identities can be shown to equal (3.8). I could not verify this equality without the aid of a symbolic mathematics computer program [48].

3.5 Appendix: The torsion of a space curve

Why is it that the relatively complicated calculation of (3.7)-(3.8) gives the relatively simple answer (3.8)? How can we be assured that when considering higher order perturbative invariants we will not get uglier formulas for $\delta\pounds_s$ for which the correcting procedure of the previous section will not work? A partial answer to these questions will be presented in this appendix — we will see that $\delta\pounds_s$ can essentially be characterized as the *only* functional that has certain invariance properties, and that these invariance properties can be deduced directly from the definition of $\delta\pounds_s$ as the variation of (3.5).

Let us start with a definition. A 1-form Ω on the space Ξ of smoothly immersed parametrized curves in $\mathbf{R}^3$ will be called a *local variation form* if it has the following properties:

1. It is *local*. Namely, if $X : S^1 \to \mathbf{R}^3$ is a smoothly immersed parametrized curve and $\omega : S^1 \to T\mathbf{R}^3 = \mathbf{R}^3$ is a tangent to Ξ, then $\Omega_X(\omega)$ is given given by the inner product of ω with a vector valued polynomial P in $|\dot{X}|^{-1}$ and finitely many derivatives of X:

   $$\Omega_X(\omega) = \int_{S^1} ds \left\langle P(|\dot{X}|^{-1}, \dot{X}, \ddot{X}, \ldots), \omega \right\rangle.$$

   The coefficients of P are, of course, expected to be independent of X and of ω. The polynomial P is uniquely determined by Ω.

2. It is *invariant* — it is independent of the parametrization of X. Namely, if $f : S^1 \to S^1$ is an orientation preserving diffeomorphism, then

   $$P(X \circ f) = \dot{f} P(X) \circ f. \tag{3.11}$$

3. It is *closed* as a 1-form on Ξ. Namely, if δ denotes exterior differentiation on Ξ, then $\delta\Omega = 0$.

4. It is $SO(3)$-invariant. Namely, if r is a rotation in $SO(3)$, then $P(r \circ X) = r \circ P(X)$.
5. It has a vanishing scaling dimension. Namely, if $C : \mathbf{R}^3 \to \mathbf{R}^3$ is the map given by multiplication by a constant c, $C(x) = cx$, then $P(C \circ X) = c^{-1}P(X)$.

It is easy to verify on *apriori* grounds that $\delta £_s$ is a local variation form — the last four properties follow immedietly from the definition of $£_s$ in (3.5), while the first property follows after a short glance at (3.7).

Theorem 1 *The form Ω^0 given by*

$$\Omega^0_X(\omega) = \int_{S^1} ds \left\langle \frac{1}{|\dot{X}|^3}\left(-3\frac{\dot{X}\cdot\ddot{X}}{|\dot{X}|^2}\dot{X}\times\ddot{X} + \dot{X}\times\dddot{X}\right), \omega\right\rangle$$

is a local variation form and every local variation form is a constant multiple thereof.

Proof The fact that Ω^0 is a local variation form follows from the fact that $\delta £_s$ is such a form, and the computation in section 3.3 which identified $\delta £_s$ to be $\Omega^0/4\pi$. The uniqueness of Ω^0 will be proven by writting the most general $SO(3)$-invariant P of vanishing scaling dimension and adjusting the coefficients so that it will be closed and parametrization independent.

By a simple enumeration, the most general $SO(3)$-invariant P of vanishing scaling dimension, which furthermore scales as (3.11) for locally constant rescallings of the parameter s is

$$\begin{aligned} P(X) \;=\; & a_1\frac{1}{|\dot{X}|^2}\dddot{X} + a_2\frac{\dot{X}\cdot\dddot{X}}{|\dot{X}|^4}\dot{X} + a_3\frac{\dot{X}\cdot\ddot{X}}{|\dot{X}|^4}\ddot{X} + a_4\frac{|\ddot{X}|^2}{|\dot{X}|^4}\dot{X} + a_5\frac{(\dot{X}\cdot\ddot{X})^2}{|\dot{X}|^6}\dot{X} \\ & + b_1\frac{1}{|\dot{X}|^3}\dot{X}\times\dddot{X} + b_2\frac{\dot{X}\cdot\ddot{X}}{|\dot{X}|^5}\dot{X}\times\ddot{X}. \end{aligned} \tag{3.12}$$

Let f be an orientation preserving diffeomorphism of R. Simple applications of the chain rule of elementary calculus yield

$$\begin{gathered} (X\circ f)' = \dot{f}\dot{X}\circ f, \qquad (X\circ f)'' = \ddot{f}\dot{X}\circ f + \dot{f}^2\ddot{X}\circ f, \\ (X\circ f)''' = \dddot{f}\dot{X}\circ f + 3\ddot{f}\dot{f}\ddot{X}\circ f + \dot{f}^3\dddot{X}\circ f. \end{gathered} \tag{3.13}$$

It is now an easy task to substitute the derivatives of $X \circ f$ into (3.12) and to look for constants a_{1-5}, $b_{1,2}$ for which the equality (3.11) holds. The result is that there are three linearly independent solutions:

$$P^0(X) = -3\frac{\dot{X}\cdot\ddot{X}}{|\dot{X}|^5}\dot{X}\times\ddot{X} + \frac{1}{|\dot{X}|^3}\dot{X}\times\dddot{X}, \tag{3.14}$$

$$P^1(X) = \frac{1}{|\dot{X}|^2}\dddot{X} - \frac{\dot{X}\cdot\dddot{X}}{|\dot{X}|^4}\dot{X} - 3\frac{\dot{X}\cdot\ddot{X}}{|\dot{X}|^4}\ddot{X} - \frac{|\ddot{X}|^2}{|\dot{X}|^4}\dot{X} + \frac{(\dot{X}\cdot\ddot{X})^2}{|\dot{X}|^6}\dot{X} \tag{3.15}$$

$$P^2(X) = \frac{|\ddot{X}|^2}{|\dot{X}|^4}\dot{X} - \frac{(\dot{X}\cdot\ddot{X})^2}{|\dot{X}|^6}\dot{X} \tag{3.16}$$

P^0 is the polynomial that gives rise to Ω^0, and we just have to prove that no other linear combination of P^0, P^1, and P^2 is closed. As P^0 is odd under a reversal of the orientation of the ambient $\mathbf{R}^3$ while P^1 and P^2 are even under such a reversal, we can restrict our attention to combinations of the form $c_1P^1 + c_2P^2$. Let us pick such a combination P^c, and let us denote the corresponding 1-form on Ξ by $\Omega^c = c_1\Omega^1 + c_2\Omega^2$. To show that Ω^c is not closed, it is enough to find two vector fields $\omega_{1,2}$ on Ξ and a point $X \in \Xi$ for which

$$\delta\Omega^c\big|_X(\omega_1, \omega_2).$$

Pick the point $X \in \Xi$ to be the unit circle in the xy-plane with its natural parametrization, and let the vector fields $\omega_{1,2}$ be given in a small neighborhood of X by two orthogonal sections of the normal bundle of X that 'rotate' around X a certain number n of times — as shown in figure 3.3. Let as now look for terms of

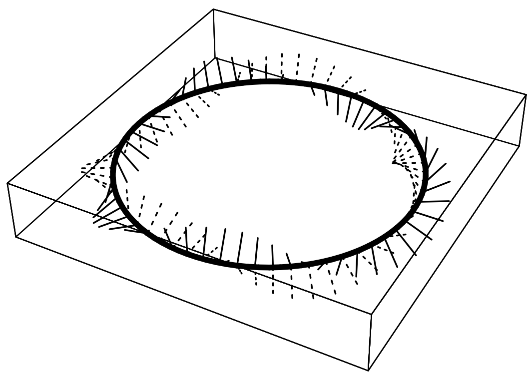


Figure 3.3. The circle X and two orthogonal infinitesimal deformations thereof that 'rotate' around it $n = 3$ times. One of the vector fields is illustrated by full lines and the other by dashed lines.

order n^3 in $\delta\Omega^c(\omega_1, \omega_2)$. As

$$\delta\Omega^c(\omega_1, \omega_2) = \int_{S^1} ds \left(\frac{\delta P^c}{\delta\omega_1} \cdot \omega_2 - \frac{\delta P^c}{\delta\omega_2} \cdot \omega_1 \right),$$

it is clear that such terms can come only from the variations of terms in P^c that involve the third derivative of X. The first such term is $c_1\dddot{X}/|\dot{X}|^2$, and its variation is

$$\begin{aligned} c_1\left(\omega_2 \cdot \frac{\delta}{\delta\omega_1} - \omega_1 \cdot \frac{\delta}{\delta\omega_2}\right)\frac{\dddot{X}}{|\dot{X}|^2} \;&=\; \frac{c_1}{|\dot{X}|^4}\Big(2(\dot{X}\cdot\dot{\omega}_2)(\dddot{X}\cdot\omega_1) - 2(\dot{X}\cdot\dot{\omega}_1)(\dddot{X}\cdot\omega_2) \\ &\qquad + |\dot{X}|^2(\dddot{\omega}_1\cdot\omega_2 - \dddot{\omega}_2\cdot\omega_1)\Big). \end{aligned}$$

Remembering that $|\dot{X}| = 1$ and that $\dddot{\omega}_1 \cdot \omega_2 = -\dddot{\omega}_2 \cdot \omega_1 \sim (const)n^3 \neq 0$, we see that the first term in P^c gives a non-vanishing contribution of order n^3 to $\delta\Omega^c$, proportional to c_1. Similarly we can compute (keeping only terms of order n^3)

$$c_1 \left(\omega_2 \cdot \frac{\delta}{\delta\omega_1} - \omega_1 \cdot \frac{\delta}{\delta\omega_2} \right) \frac{\dot{X} \cdot \dddot{X}}{|\dot{X}|^4} \dot{X} \sim c_1 \left((\dot{X} \cdot \dddot{\omega}_1)(\dot{X} \cdot \omega_2) - (\dot{X} \cdot \dddot{\omega}_2)(\dot{X} \cdot \omega_1) \right) = 0$$

using the orthogonality of $\dot{X}$ and $\omega_{1,2}$.

Therefore, in order to have $\delta\Omega^c(\omega_1, \omega_2) = 0$ we must have $c_1 = 0$, namely $\Omega^c = c_2\Omega^2$. Computations of exactly the same nature as the above now show that in order to have no term proportional to n in $\delta\Omega^c(\omega_1, \omega_2)$ we must have $c_2 = 0$.

□

Remark The above theorem and the results of the previous section combine into a somewhat amusing property of the total torsion of a space curve — it can be represented as an integral of a local quantity (3.10), but not in a canonical way ((3.10) depends on the non canonical choice of n, and (3.9) is ill defined for some curves). Yet its variation $\delta\tau = -\delta\pounds_s = -i\Omega^0/2$ can be represented canonically as the integral of a local quantity, and *it is the only global quantity (of vanishing scaling dimension) whose variation can be represented in a parametrization independent manner.*

Remark The only direct proof I know for the crucial equality $\delta\tau = -i\Omega^0/2$ is described in the paragraph proceeding (3.10). This proof is very tedious and uses a computer for some of the algebra involved. However, there are simple arguments that establish directly that $\delta\tau = -\delta\pounds_s$ (see e.g. [34]). In section 3.3 we saw that $\delta\pounds_s = i\Omega^0/2$, and the last two facts together constitute a reasonably simple proof of the equality $\delta\tau = -i\Omega^0/2$.

Remark In terms of the Frenet frame (T, N, B) (see [18]), we have

$$P^0(X) = \dot{\kappa}B - \kappa\tau N.$$

Chapter 4

The two-loop contribution

4.1 Statement of the problem

Let X be a parametrized knot in $\mathbf{R}^3$. In this chapter we will try to understand the two-loop contribution $\mathcal{W}_2$ to $\mathcal{W}(\text{flat } \mathbf{R}^3, \mathcal{X})$ — the contribution of order $-4\pi^2/k^2$. All the terms in the Lagrangian $\mathcal{L}_{tot}$ come in to play now, and on a flat $\mathbf{R}^3$ our $\mathcal{W}$ reads

$$\mathcal{W}(\text{flat } \mathbf{R}^3, \mathcal{X}) = \int \mathcal{D}A\mathcal{D}\phi\mathcal{D}c\mathcal{D}\bar{c}\ tr_R\mathcal{P}\exp\left(\int ds\dot{X}^i(s)A_i(X(s))\right) e^{i\mathcal{L}_{tot}}$$

where

$$\mathcal{L}_{tot} = \frac{k}{4\pi}\int_{\mathbf{R}^3} \mathfrak{tr}\left(\epsilon^{ijk}A_i\partial_jA_k + 2\phi\partial^iA_i + \frac{1}{3}\epsilon^{ijk}A_i[A_j, A_k] + 2\bar{c}\partial_i(\partial^ic + [A^i, c])\right)$$

If R is a unimodular[1] representation, terms that have only one interaction point with X have a vanishing coefficient and therefore the only potential contribution at two-loops come from the five diagrams in figure 4.1.

The first two diagrams are divergent because of the integration over the location of the interaction vertices in $\mathbf{R}^3$. But as is readily verified and as was shown in [24] the integrands in these diagrams are exactly the opposites of each other so if we sum them before integrating we get zero. (We will accept at face value that A and B cancel and prove that $C + D + E$ is a topological invariant. It is very likely that the full story is a little more elaborate. In the context of a consistent regularization that could be used to all orders, A and B are likely to cancel only up to an imaginary multiple of the one loop contribution and thus what is calculated here is just the *real* part of the two-loop contribution. See chapter 5 and [33, 2, 16]). Also, it is clear that if one ignores the Lie-algebra coefficients of diagrams C and D and the combinatorial coefficients $S(C)$ and $S(D)$ then their sum is equal to the square of the one-loop one-knot contribution that was discussed in the previous chapter. It is therefore possible to subtract from $\mathcal{W}_2$ a multiple of $(\mathcal{W}_1)^2$ in such a way that diagram C will disappear. We will call the resulting quantity $\hat{\mathcal{W}}_2$. The coefficient of diagram D in $\hat{\mathcal{W}}_2$ will be the

[1] Namely, a representation by linear operators of trace zero.

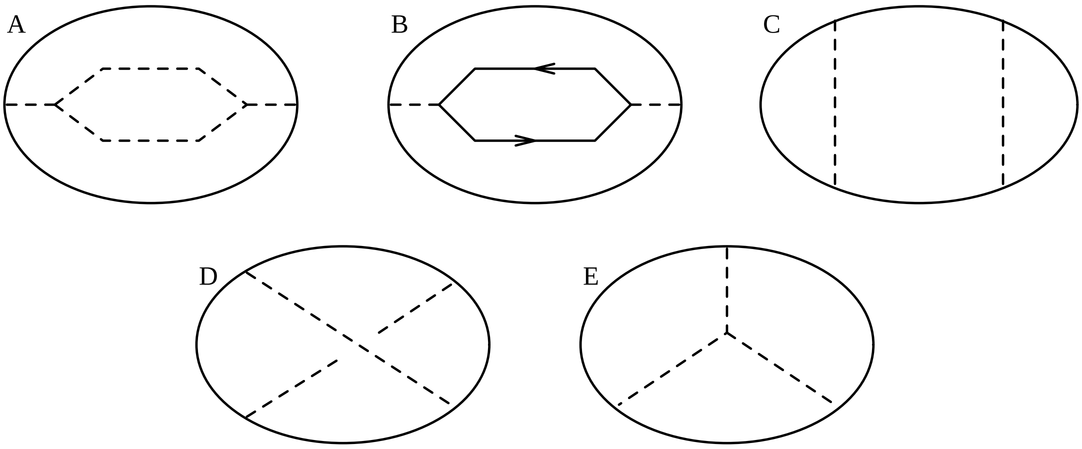


Figure 4.1. The five two-loop diagrams.

difference between the coefficients of diagrams D and C in $\mathcal{W}_2$, and these coefficients differ only because the Lie-Algebra indices are contracted in a slightly different way. So if $t_{ab} \stackrel{\text{def}}{=} \mathrm{tr}(\mathcal{G}_a\mathcal{G}_b)$, t^{ab} is the inverse matrix of t_{ab} and we use t_{ab} and t^{ab} to raise and lower Lie-algebra indices, we get[2]:

$$C(D) - C(C) \stackrel{\text{def}}{=} \begin{pmatrix}\text{Lie algebra con-}\\ \text{tractions for } D\end{pmatrix} - \begin{pmatrix}\text{Lie algebra con-}\\ \text{tractions for } C\end{pmatrix}$$

$$= t^{bb'}t^{cc'}R^{\beta}_{b'\delta}R^{\delta}_{c'\gamma}R^{\gamma}_{b\alpha}R^{\alpha}_{c\beta} - t^{bb'}t^{cc'}R^{\beta}_{c'\delta}R^{\delta}_{b'\gamma}R^{\gamma}_{b\alpha}R^{\alpha}_{c\beta} \tag{4.1}$$

The fact that R is a representation is just the relation $-f^{abc}R^{\beta}_{a\gamma} = t^{bb'}t^{cc'}(R^{\beta}_{b'\delta}R^{\delta}_{c'\gamma} - R^{\beta}_{c'\delta}R^{\delta}_{b'\gamma})$ and therefore

$$(4.1) = -f^{abc}R^{\beta}_{a\gamma}R^{\gamma}_{b\alpha}R^{\alpha}_{c\beta} \stackrel{\text{def}}{=} -C(E).$$

These are exactly the negatives of the Lie-algebra contractions for diagram E. Taking into account the different symmetry factors for these diagrams we finally get (after dividing by the Lie algebraic coefficient)

$$\tilde{\mathcal{W}}_2 \stackrel{\text{def}}{=} \frac{1}{C(E)}\hat{\mathcal{W}}_2 = -\frac{1}{4}\int\mathcal{E}(D) + \frac{1}{3}\int\mathcal{E}(E).$$

More explicitly, if diagrams D and E are marked as in figure 4.2, then $\tilde{\mathcal{W}}_2$ is given by

$$\tilde{\mathcal{W}}_2 = \frac{1}{16}\int_{\Delta_4} ds_{1-4}\dot{X}^i_1\dot{X}^j_2\dot{X}^k_3\dot{X}^l_4\epsilon_{ikm}\epsilon_{jln}\frac{(X_1-X_3)^m}{|X_1-X_3|^3}\frac{(X_2-X_4)^n}{|X_2-X_4|^3}$$

[2]The Lie-algebra computation below is a particular case of the "*STU*" relation of chapter 9.

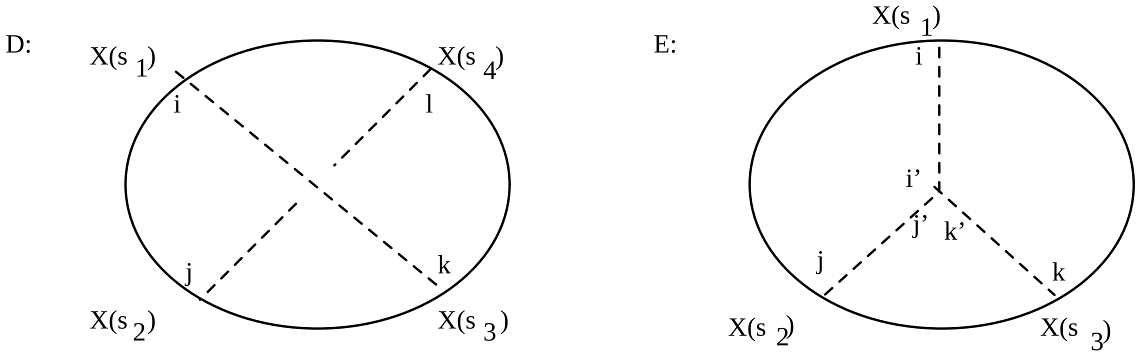


Figure 4.2. The two contributing diagrams.

$$-\frac{1}{48\pi}\int_{\Delta_3} ds_{1,2,3}\int_{\mathbf{R}^3} d^3z\,\dot{X}_1^i\dot{X}_2^j\dot{X}_3^k\epsilon^{i'j'k'}\epsilon_{ii'i''}\epsilon_{jj'j''}\epsilon_{kk'k''}$$
$$\frac{(X_1-z)^{i''}}{|X_1-z|^3}\frac{(X_2-z)^{j''}}{|X_2-z|^3}\frac{(X_3-z)^{k''}}{|X_3-z|^3}, \tag{4.2}$$

where X_i stands for $X(s_i)$, $i = 1, \ldots, 4$.

In the case of $G = SU(N)$; $R = \mathbf{C}^N$ one can calculate[3] that in $\mathcal{W}_2$ the Lie-algebraic coefficients of diagrams C, D, and E are $\frac{(N^2-1)^2}{N}$, $\frac{1-N^2}{N}$, and $N(N^2-1)$ respectively, and therefore in this case

$$\tilde{\mathcal{W}}_2 = \frac{1}{N(N^2-1)}\left(\mathcal{W}_2 - \frac{1}{2N}(\mathcal{W}_1)^2\right).$$

Our aim in the rest of this chapter is to prove the following theorem:

Theorem 2 *Let X be a parametrized knot in $\mathbf{R}^3$. (that is to say, X is a smooth non-singular function from S^1 to $\mathbf{R}^3$ that has no self intersections). Then the integrals represented by the diagrams D and E of figure 4.2 are convergent, and their sum $\tilde{\mathcal{W}}_2$ is an isotopy invariant of the knot X. This invariant can be identified to be $-\pi^2/6$ minus $4\pi^2$ times the second non trivial coefficient of the Conway polynomial of X, whose reduction mod 2 is the well known Arf invariant of X.*

4.2 The finiteness of $\tilde{\mathcal{W}}_2$

It still isn't clear that the integrals represented by the diagrams D and E are finite. For diagram D there appears to be a singularity when three of the integration variables are close together but exactly the same analysis that has shown that the self-linking integral is finite shows that this integral is also finite. In diagram E there appears to

[3]See chapter 9 for the details.

be a problem when two or three of the knot integration variables are close together and are close to z — the variable of the A^3 vertex integration. Up to a constant factor, diagram E represents the integral:

$$\int \mathcal{E}(E) = \int_{\Delta_3} ds_{1,2,3} \dot{X}^i(s_1)\dot{X}^j(s_2)\dot{X}^k(s_3) V_{ijk}(X(s_1), X(s_2), X(s_3)) \tag{4.3}$$

where

$$V_{ijk}(x_1,x_2,x_3) \stackrel{\text{def}}{=} \epsilon^{i'j'k'}\epsilon_{ii'i''}\epsilon_{jj'j''}\epsilon_{kk'k''}T^{i''j''k''}(x_1,x_2,x_3) \stackrel{\text{def}}{=} 6_{ijki''j''k''}T^{i''j''k''}(x_1,x_2,x_3)$$

and

$$T^{ijk}(x_1,x_2,x_3) \stackrel{\text{def}}{=} \int_{\mathbf{R}^3} d^3z \frac{(x_1-z)^i}{|x_1-z|^3}\frac{(x_2-z)^j}{|x_2-z|^3}\frac{(x_3-z)^k}{|x_3-z|^3}$$

The integral defining T is clearly finite for every choice of distinct x_{1-3} in $\mathbf{R}^3$, but it blows up rapidly when some of the x's coincide. To show that in spite of this the integral (4.3) is finite we need to understand the behavior of T as two or three of its arguments coincide.

4.2.1 A simpler expression for T

Let us first rewrite T in a way that will make it easier to handle. Using

$$\frac{4}{3\sqrt{\pi}}\int_0^\infty e^{-\alpha^{2/3}N}d\alpha = \frac{1}{N^{3/2}}$$

we can rewrite T as

$$T^{ijk} = \frac{64}{27\pi^{3/2}}\int_0^\infty d^3\alpha \int_{\mathbf{R}^3} dz (x_1-z)^i(x_2-z)^j(x_3-z)^k e^{-\sum_{m=1}^3 \alpha_m^{2/3}|x_m-z|^2}.$$

Introducing the notation:

$$\begin{array}{rclcrcl} A &=& \sum \alpha_m^{2/3} & ; & \lambda_m &=& \dfrac{\alpha_m^{2/3}}{A} \\ t &=& \sum \lambda_m x_m & ; & s &=& \sum \lambda_m |x_m - t|^2 \end{array}$$

we get

$$\begin{aligned} T^{ijk}(x_1,x_2,x_3) &= \frac{64}{27\pi^{3/2}}\int_0^\infty d^3\alpha \int_{\mathbf{R}^3} dz (x_1-z)^i(x_2-z)^j(x_3-z)^k e^{-A(|z-t|^2+s)} \\ &= \frac{64}{27\pi^{3/2}}\int_0^\infty d^3\alpha e^{-As} \int_{\mathbf{R}^3} dz (x_1-t-z)^i(x_2-t-z)^j(x_3-t-z)^k e^{-A|z|^2}. \end{aligned}$$

This is just a Gaussian integral with respect to z, and it can be evaluated to give

$$\begin{aligned} T^{ijk} &= \frac{64}{27}\int_0^\infty d^3\alpha \frac{e^{-As}}{A^{3/2}}\left[\frac{1}{2A}\left((x_1-t)^i\delta^{jk} + (x_2-t)^j\delta^{ki} + (x_3-t)^k\delta^{ij}\right)\right. \\ &\qquad \left. + (x_1-t)^i(x_2-t)^j(x_3-t)^k\right]. \end{aligned}$$

Changing variables from $d^3\alpha$ to $d^2\lambda dA$ (there are just two integrations over the λ's because they are constrained to satisfy $\sum \lambda_m = 1$) we pick the Jacobian $\frac{27}{8}A^{7/2}\sqrt{\lambda_1\lambda_2\lambda_3}$ and get (after evaluating the A integral)

$$\begin{aligned} T^{ijk}(x_1,x_2,x_3) &= 4\int d^2\lambda\sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{(x_1-t)^i\delta^{jk}+(x_2-t)^j\delta^{ki}+(x_3-t)^k\delta^{ij}}{s^2}\right. \\ &\qquad \left.+4\frac{(x_1-t)^i(x_2-t)^j(x_3-t)^k}{s^3}\right]. \end{aligned} \tag{4.4}$$

4.2.2 Bounding the possible divergence

Clearly the integral (4.3) is translation invariant, and invariant under reparametrizations of X of the form $s \to s + s_0$. So in the investigation of its possible divergencies we can assume that, say, 0 is the midpoint between s_2 and s_3, s_1 is farther away from s_2 or s_3 than the distance between these two:

$$s_1 = \tau \qquad ; \qquad s_2 = -\eta\tau \qquad ; \qquad s_3 = \eta\tau \qquad ; \qquad |\eta| < \frac{1}{3},$$

and that $X(0) = 0$. In this case we can write

$$T^{ijk}(X_\tau, X_{-\eta\tau}, X_{\eta\tau}) = 4\int d^2\lambda\sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{S_1^{ijk}}{s^2} + 4\frac{S_2^{ijk}}{s^3}\right] \tag{4.5}$$

with

$$\begin{aligned} S_1^{ijk} &\stackrel{\text{def}}{=} (X_\tau - t)^i\delta^{jk} + (X_{-\eta\tau} - t)^j\delta^{ki} + (X_{\eta\tau} - t)^k\delta^{ij}, \\ S_2^{ijk} &\stackrel{\text{def}}{=} (X_\tau - t)^i(X_{-\eta\tau} - t)^j(X_{\eta\tau} - t)^k. \end{aligned}$$

The problematic regions are when η or τ are small, and we need to be able to estimate integrals like those in (4.5) for such values of η and τ.

Lemma 4.2.1 *Let A, B, and C be the three vertices of a triangle with sides $|A-B| \sim |A-C| \sim \tau$, and $|B-C| \sim \eta\tau$ with $\eta < 1/3$ (see figure 4.3). For positive λ's satisfying $\lambda_1 + \lambda_2 + \lambda_3 = 1$ define:*

$$\begin{aligned} t &= \lambda_1 A + \lambda_2 B + \lambda_3 C \\ s &= \lambda_1|A-t|^2 + \lambda_2|B-t|^2 + \lambda_3|C-t|^2 \end{aligned}$$

Finally let Λ_A be one of $\{(1-\lambda_1), \lambda_2, \lambda_3\}$, Λ_B be one of $\{\lambda_1,\ (1-\lambda_2), \lambda_3\}$, and Λ_C be one of $\{\lambda_1, \lambda_2, (1-\lambda_3)\}$.
In this situation there exists constants c_{1-5} independent of η and τ for which:

$$\int d^2\lambda\sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{1}{s^2}\right] < \frac{c_1}{\eta\tau^4} \tag{4.6}$$

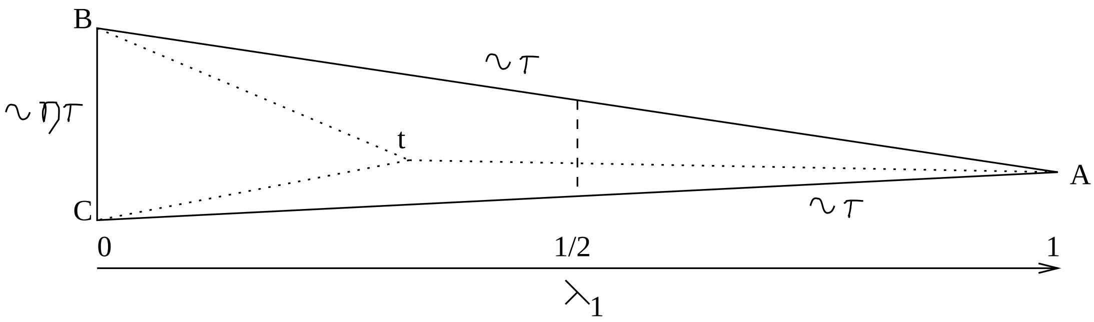


Figure 4.3. The triangle ABC.

$$\int d^2\lambda \sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{\lambda_1}{s^2}\right] \quad < \quad \frac{c_2}{\tau^4} \tag{4.7}$$

$$\int d^2\lambda \sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{\Lambda_A\Lambda_B\Lambda_C}{s^3}\right] \quad < \quad \begin{cases} \dfrac{c_3}{\eta^3\tau^6} & \textit{if neither of } \Lambda_B \textit{ or } \Lambda_C \textit{ is chosen to be } \lambda_1, \\[2ex] \dfrac{c_4}{\eta\tau^6} & \textit{if exactly one of } \Lambda_B,\ \Lambda_C \textit{ is chosen to be } \lambda_1, \\[2ex] \dfrac{c_5}{\tau^6} & \textit{if both of } \Lambda_B \textit{ and } \Lambda_C \textit{ are chosen to be } \lambda_1. \end{cases} \tag{4.8}$$

Proof We will write $\lambda_2 = (1-\lambda_1)\theta$ and $\lambda_3 = (1-\lambda_1)\bar{\theta}$ where $0 \le \theta \le 1$ and $\bar{\theta} = 1-\theta$. c will denote a positive constant that is allowed to change from line to line. It is easy to read from the geometry of figure 4.3 that when $\lambda_1 < 1/2$, (equivalently, when t is in the left portion of figure 4.3)

$$s|_{\lambda_1<\frac{1}{2}} > c\left(\theta\bar{\theta}^2|B-C|^2 + \bar{\theta}\theta^2|B-C|^2 + \lambda_1\tau^2\right) > c\tau^2\left(\theta\bar{\theta}\eta^2 + \lambda_1\right). \tag{4.9}$$

Also, it is clear that the major contribution to (4.6), (4.7), and (4.8) comes from that region when $\lambda_1 < 1/2$, and therefore (4.9) can be used to give upper bounds for the integrals we are considering.

Taking for example (4.8) with $\Lambda_A = (1-\lambda_1)$, $\Lambda_B = (1-\lambda_2)$, and $\Lambda_C = (1-\lambda_3)$ we get

$$\int d^2\lambda \sqrt{\lambda_1\lambda_2\lambda_3}\left[\frac{\Lambda_A\Lambda_B\Lambda_C}{s^3}\right] < c\int_0^1 d\theta \int_0^{\frac{1}{2}} d\lambda_1 \frac{\sqrt{\lambda_1\theta\bar{\theta}}(\lambda_1+\theta\bar{\theta})}{\tau^6\left(\theta\bar{\theta}\eta^2+\lambda_1\right)^3}. \tag{4.10}$$

The λ_1 integral can be explicitly evaluated. In fact, for a small α one has

$$\int_0^a d\lambda \frac{\sqrt{\lambda}}{(\alpha^2+\lambda)^3} = \frac{-\sqrt{a}}{2(a+\alpha^2)^2} + \frac{\sqrt{a}}{4\alpha^2\,(a+\alpha^2)} + \frac{\arctan(\frac{\sqrt{a}}{\alpha})}{4\alpha^3} < \frac{c}{\alpha^3}$$

and

$$\int_0^a d\lambda \frac{\sqrt{\lambda}\lambda}{(\alpha^2+\lambda)^3} = \frac{\sqrt{a}\alpha^2}{2(a+\alpha^2)^2} - \frac{5\sqrt{a}}{4\,(a+\alpha^2)} + \frac{3\arctan(\frac{\sqrt{a}}{\alpha})}{4\alpha} < \frac{c}{\alpha}$$

and plugging these two estimates into (4.10) gives the required result. The other assertions of the lemma are proved along the same lines.

□

4.2.3 Proof of the finiteness of diagram E

It is sufficient to show that

$$T^{ijk}(X_\tau, X_{-\eta\tau}, X_{\eta\tau}) < c/\tau. \tag{4.11}$$

Let us first deal with the contribution coming from S_1^{ijk}. Expanding S_1^{ijk} in powers of λ_1,

$$S_1^{ijk} = S_1^{0,ijk} + \lambda_1 S_1^{1,ijk} \tag{4.12}$$

we can use (4.6) and (4.7) and then all that is left to prove is:

$$6_{ijki'j'k'}\dot{X}^{i'}(\tau)\dot{X}^{j'}(-\eta\tau)\dot{X}^{k'}(\eta\tau)S_1^{p,ijk} = O(\eta^{1-p}\tau^3) \qquad ; \qquad p=0,1. \tag{4.13}$$

This can be done by expanding all the terms in the above expressions once in powers of η and once in powers of τ and showing that the low order coefficients in each of these expansions are zero. It is not hard to do it by hand, but as we are going to encounter some very similar but a bit harder expansions later on we will not do it here but postpone it to the appendix where it will be shown how all these expansions can be carried out in a uniform way using a computer.

The terms involving S_2^{ijk} are dealt with in a very similar way. Clearly, each of the factors of S_2^{ijk} is made of three summands, whose coefficients exactly correspond to the various possibilities for choosing Λ_A, Λ_B, and Λ_C in the lemma 4.2.1. Keeping $(X_\tau - t)^i$ unexpanded and expanding *only* the last two factors of S_2^{ijk} in powers of λ_1,

$$S_2^{ijk} = S_2^{0,ijk} + \lambda_1 S_2^{1,ijk} + \lambda_1^2 S_2^{2,ijk}, \tag{4.14}$$

and keeping in mind (4.8) what is left to prove is

$$6_{ijki'j'k'}\dot{X}^{i'}(\tau)\dot{X}^{j'}(-\eta\tau)\dot{X}^{k'}(\eta\tau)S_2^{p,ijk} = \begin{cases} O(\eta^3\tau^5) & \text{for } p=0, \\ O(\eta\tau^5) & \text{for } p=1, \\ O(\tau^5) & \text{for } p=2. \end{cases} \tag{4.15}$$

Again, the relevant expansions will be shown to vanish to the required order in the appendix using a computer.

4.3 The invariance of $\tilde{\mathcal{W}}_2$

4.3.1 The regularized $\tilde{\mathcal{W}}_2$

We will now show that $\tilde{\mathcal{W}}_2$ is indeed a knot invariant — that it is not changed under infinitesimal deformations. The proof presented here should be similar in spirit to invariance proofs (that are yet to be found) of higher terms in the perturbative expansion — we will first write a diagrammatic argument, and then supplement it with the required analytical details. As in the case of the analysis of the variation of the self linking number in the previous chapter, in analyzing the variation of $\tilde{\mathcal{W}}_2$ we will need take derivatives of V_{ijk} and of V_{ij} near the diagonal where there are singularities which will prevent a straight-forward invariance proof. To avoid these singular points define $\tilde{\mathcal{W}}_{2,\epsilon}$ to be given by the same integrals $\int\mathcal{E}(D)$ and $\int\mathcal{E}(E)$ as $\tilde{\mathcal{W}}_2$, only with the integration domain restricted by the condition that the s's would be at least ϵ apart — for $i \neq j$ we require

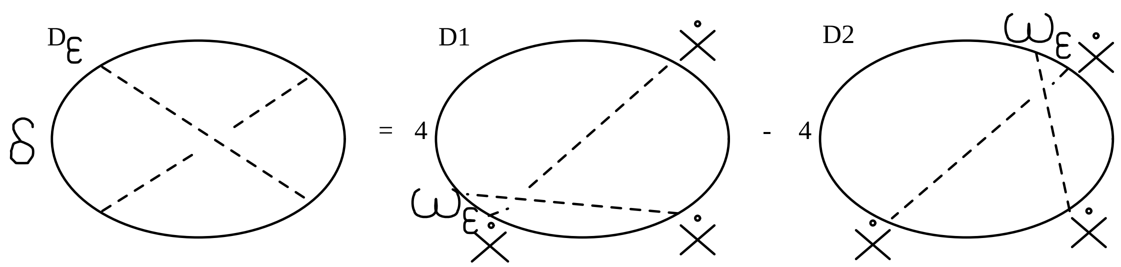
$$|s_i - s_j| > \epsilon. \tag{4.16}$$

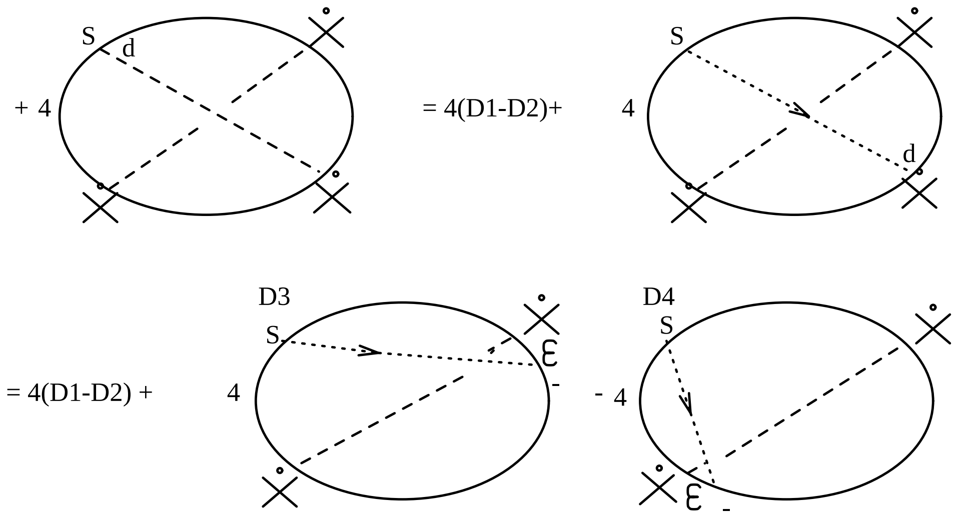
We will denote these integrals by D_ϵ and E_ϵ, and the finiteness of $\tilde{\mathcal{W}}_2$ that was proven above just means

$$\tilde{\mathcal{W}}_{2,\epsilon} = -\frac{1}{4}D_\epsilon + \frac{1}{3}E_\epsilon \xrightarrow[\epsilon\to 0]{} -\frac{1}{4}\int\mathcal{E}(D) + \frac{1}{3}\int\mathcal{E}(E) = \tilde{\mathcal{W}}_2. \tag{4.17}$$

4.3.2 The variation of $\tilde{\mathcal{W}}_2$

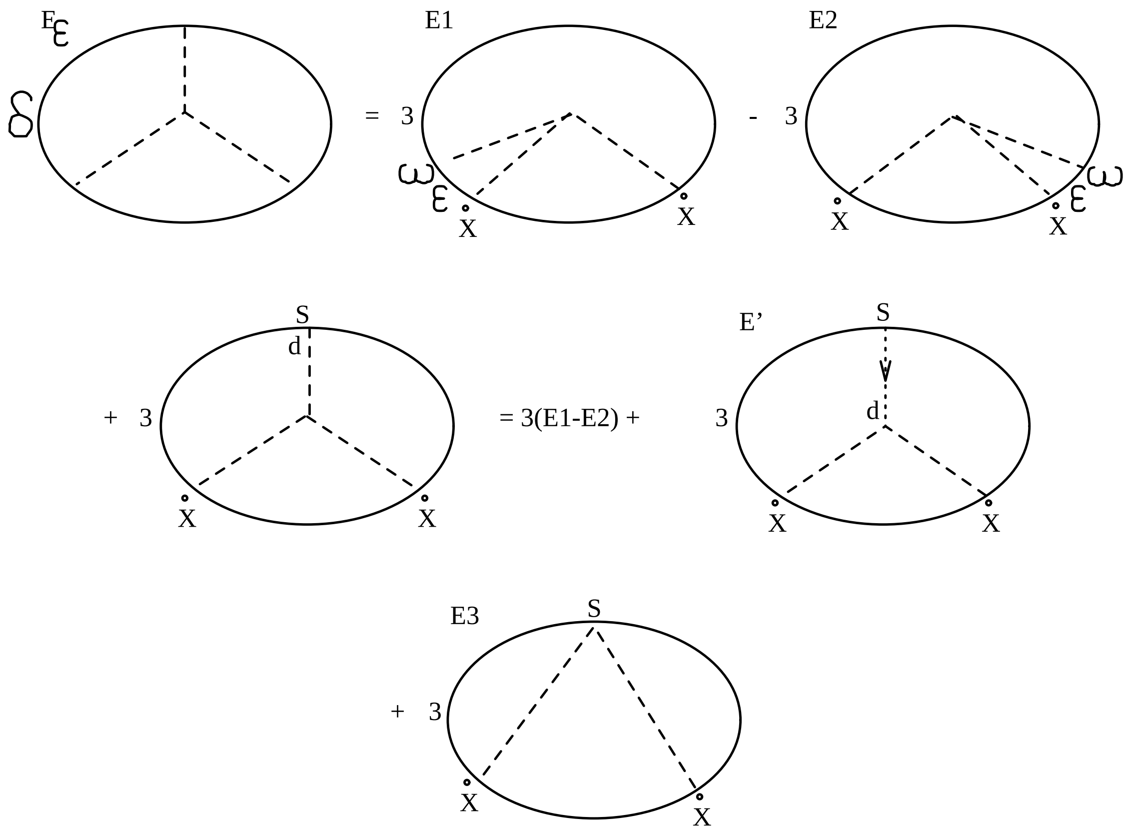
We will now vary D_ϵ and E_ϵ under an infinitesimal deformation of X given by $X \to X+\omega$. It will be a lot more instructive to perform those calculations diagrammatically instead of working with the explicit formulae given for D and E in (4.2). First, let us vary D_ϵ. When X moves to $X + \omega$ it swaps an infinitesimal surface S, and our quantity of interest δD_ϵ is given by the evaluation of $d^L V$ on S which after using the key relation (3.2) reduces to diagrams $D3$ and $D4$ and by two boundary terms, diagrams $D1$ and $D2$:



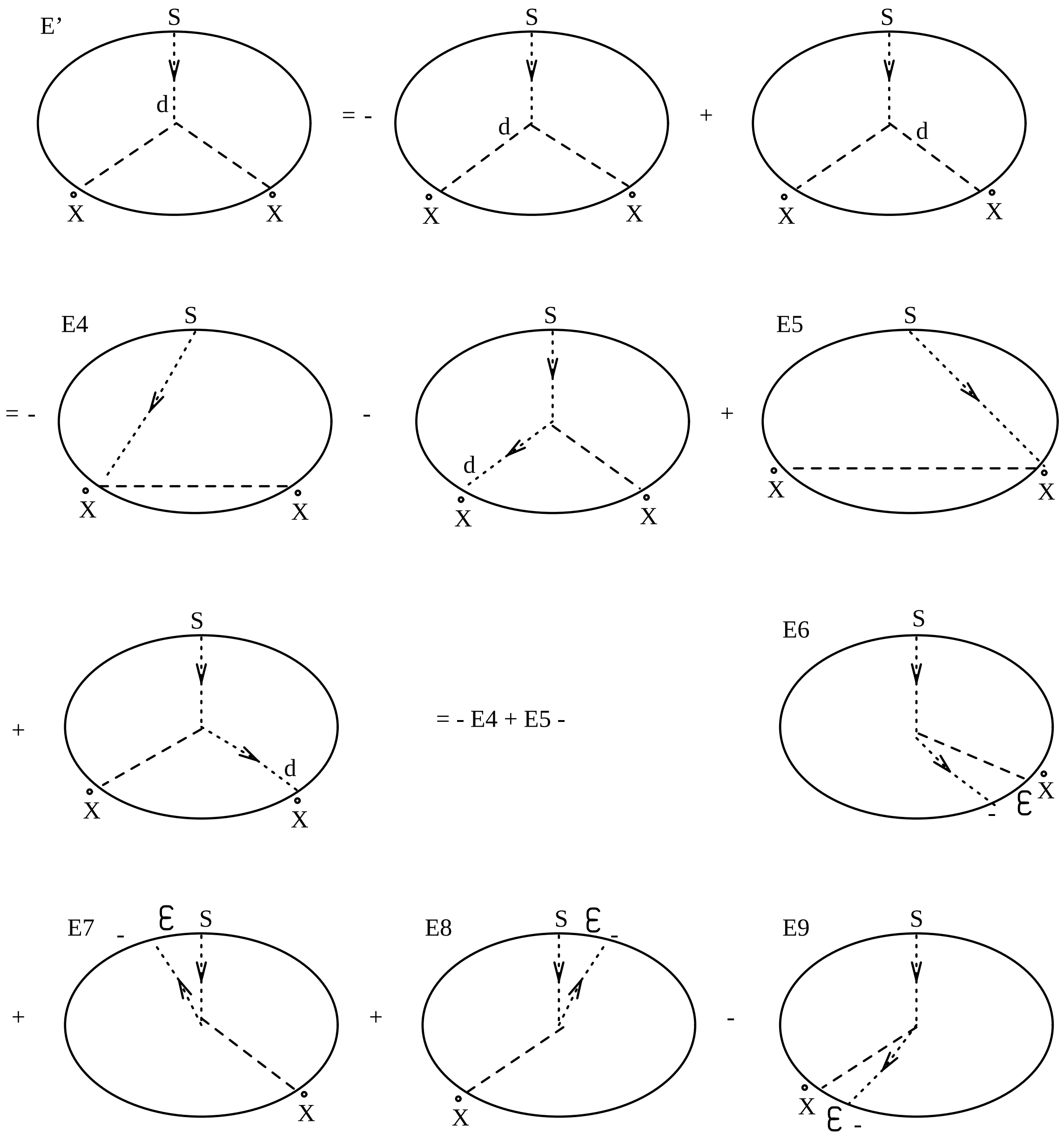


In these diagrams a dashed line represents as before the gauge propagator V_{ij} evaluated between the two vectors marked at its ends, a dotted represents the $(2,0)$-form F, a d symbol stands for exterior differentiation applied to the nearby end of the nearby propagator, and an ϵ between two interaction points on the knot means that these points are exactly ϵ apart.

Similarly we can vary E_ϵ:



The diagram $E3$ appears because (3.2) is true only *off* diagonal. Actually $d^L V$ and $-d^R F$ differ by a $\star^L$ of a δ-function as was shown in the derivation of (3.2). Integrating by parts and using Leibnitz's rule we get:



4.3.3 The invariance proof

To show that $\tilde{\mathcal{W}}_2$ is indeed an invariant we first need to show that the limit as $\epsilon \to 0$ of $\delta(-\frac{1}{4}D_\epsilon + \frac{1}{3}E_\epsilon)$ vanishes. That is, we need to show that

$$\lim_{\epsilon \to 0} -D1 + D2 - D3 + D4 + E1 - E2 + E3 - E4 + E5 - E6 + E7 + E8 - E9 = 0.$$

In fact, we will show that

$$\lim_{\epsilon\to 0} -D1 + D2 + E3 = 0, \tag{4.18}$$

$$\lim_{\epsilon\to 0} -D3 + D4 - E4 + E5 = 0, \tag{4.19}$$

and

$$\lim_{\epsilon\to 0} E1 - E2 - E6 + E7 + E8 - E9 = 0, \tag{4.20}$$

independently. For convenience, the symbol $\int_\epsilon$ will denote integration in which the integration variables are constrained to satisfy the restrictions (4.16), we will write X_ν for $X(s_\nu)$, and similarly for $\dot{X}_\nu$, $\ddot{X}_\nu$ and ω_ν.

Proof of (4.18) Diagram $D1$ represents the integral

$$-D1 = -\int_\epsilon ds_{1-3}\omega_3^i \dot{X}_4^k V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_4, X_2)\dot{X}_2^l \qquad ; \qquad s_4 = s_3 + \epsilon, \tag{4.21}$$

diagram $D2$ reads

$$D2 = \int_\epsilon ds_{1-3}\dot{X}_3^i \omega_4^k V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_4, X_2)\dot{X}_2^l \qquad ; \qquad s_4 = s_3 + \epsilon, \tag{4.22}$$

and diagram $E3$ is given by

$$E3 = -\int_\epsilon ds_{1-3}\dot{X}_3^p \omega_3^n \epsilon_{pnm}\epsilon^{mik} V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_3, X_2)\dot{X}_2^l. \tag{4.23}$$

Using

$$\epsilon_{pnm}\epsilon^{mik} = \delta_p^i\delta_n^k - \delta_p^k\delta_n^i$$

we can write $E3 = E3' + E3''$ with

$$E3' = -\int_\epsilon ds_{1-3}\dot{X}_3^i \omega_3^k V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_3, X_2)\dot{X}_2^l. \tag{4.24}$$

and

$$E3'' = \int_\epsilon ds_{1-3}\dot{X}_3^k \omega_3^i V_{ij}(X_3, X_1)\dot{X}_1^j V_{kl}(X_3, X_2)\dot{X}_2^l.. \tag{4.25}$$

The nearness of s_3 and s_4 clearly implies that the integrand in (4.21) converges to the integrand of (4.25) and the integrand in (4.22) converges to the integrand of (4.24) as $\epsilon \to 0$. At the region where s_1 and s_2 are farther from $s_{3,4}$ than some fixed but small positive constant T, there is no problem with commuting integration with taking the $\epsilon \to 0$ limit. Concentrating first on comparing diagrams $D1$ and $E3''$ we see that nothing particularly harmful happens if *just* $|s_4 - s_2|$ is small — as it was shown in chapter 3 the integrand in this case remains finite. Otherwise, we are looking at one of the following exceptional cases (assuming for simplicity that $s_4 = 0$, $s_3 = -\epsilon$, and $X_4 = 0$):

Case 1: Disregarding the propagator connecting X_2 and $X_4 = 0$ the difference $-D1 + E3''$ reads:

$$\int_\epsilon^T ds_1 \frac{\det\left(\omega(-\epsilon)\left|\dot{X}_1\right| X(-\epsilon) - X_1\right)}{|X(-\epsilon) - X_1|^3} - \int_\epsilon^T ds_1 \frac{\det\left(\omega(0)\left|\dot{X}_1\right| X(0) - X_1\right)}{|X(0) - X_1|^3}. \tag{4.26}$$

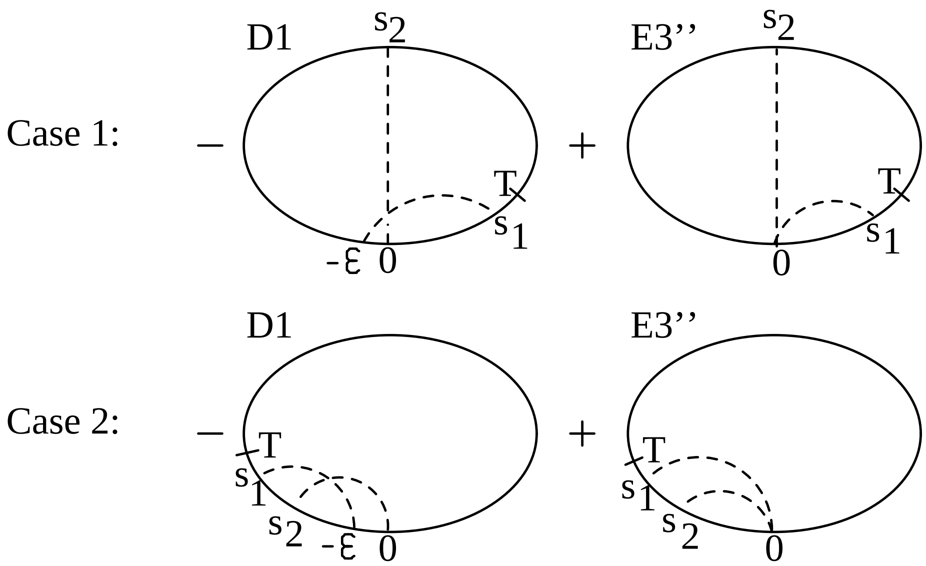


Figure 4.4. The two exceptional cases for $D1 \leftrightarrow E3''$.

Expanding the integrands in (4.26) in powers of s_1 we can ignore all terms of order smaller than $1/s_1$ — evaluating the integrals in (4.26) for these terms would give a result bounded by a constant multiple of T in the $\epsilon \to 0$ limit, and as T was chosen small we can indeed ignore the contribution to (4.26) coming from these terms. There are no terms of order *higher* than $1/s_1$ in (4.26) and the term of order $1/s_1$ reads:

$$\int_\epsilon^T ds_1 \left(\frac{1}{2(s_1+\epsilon)} \frac{\det\left(\omega(-\epsilon)\left|\dot{X}(-\epsilon)\right|\ddot{X}(-\epsilon)\right)}{|\dot{X}(-\epsilon)|^3} - \frac{1}{2s_1} \frac{\det\left(\omega(0)\left|\dot{X}(0)\right|\ddot{X}(0)\right)}{|\dot{X}(0)|^3} \right)$$

at the $\epsilon \to 0$ limit we get

$$\sim \frac{\det\left(\omega(0)\left|\dot{X}(0)\right|\ddot{X}(0)\right)}{2|\dot{X}(0)|^3} \int_\epsilon^T ds_1 \left(\frac{1}{s_1+\epsilon} - \frac{1}{s_1} \right) \to -\frac{\det\left(\omega(0)\left|\dot{X}(0)\right|\ddot{X}(0)\right)\log 2}{2|\dot{X}(0)|^3}. \tag{4.27}$$

Reinstalling the propagator connecting X_2 and X_4 and the integration over s_2 we get the only non-vanishing contribution to $-D1 + E3''$.

Case 2: Here the $\epsilon \to 0$ limit is in fact zero. To see that, one does analysis similar to the previous case, and notices that s_2 is integrated over an interval of length smaller than s_1 and thus remembering that the propagator connecting X_2 and X_4 is finite even near the diagonal the s_2 integral is $\sim s_1$, and this additional factor is sufficient to make the contribution coming from this case vanish.

A similar analysis to the above shows that the only non-vanishing contribution to $D2 - E3'$ comes from the case parallel to case 1 here, and that, in fact, these contributions exactly cancel.

$\square$

Proof of (4.19) Here are the integrals corresponding to the relevant diagrams:

$$
\begin{aligned}
-D3 &= -\int_\epsilon ds_{1-3}\dot{X}_4^k V_{lk}(X_2,X_4)\dot{X}_2^l\dot{X}_1^i\omega_1^j F_{ij,-}(X_1,X_3), \quad ; \quad s_4 = s_3+\epsilon, && (4.28)\\
D4 &= \int_\epsilon ds_{1-3}\dot{X}_3^k V_{kl}(X_3,X_1)\dot{X}_1^l\dot{X}_2^i\omega_2^j F_{ij,-}(X_2,X_4), \quad ; \quad s_4 = s_3+\epsilon, && (4.29)\\
-E4 &= \frac{1}{2}\int_\epsilon ds_{1-3}\dot{X}_3^k\epsilon_{kmn}\epsilon^{mnp}V_{pl}(X_3,X_1)\dot{X}_1^l F_{ij,-}(X_2,X_3)\dot{X}_2^i\omega_2^j, && (4.30)\\
E5 &= -\frac{1}{2}\int_\epsilon ds_{1-3}\dot{X}_3^k\epsilon_{kmn}\epsilon^{mnp}V_{pl}(X_3,X_2)\dot{X}_2^l F_{ij,-}(X_1,X_3)\dot{X}_1^i\omega_1^j. && (4.31)
\end{aligned}
$$

Using

$$\epsilon_{kmn}\epsilon^{mnp} = 2\delta_k^{\ p}$$

and the nearness of s_3 and s_4 it is clear that so long as X_1 and X_2 are far away from X_3 the integrands of (4.28) and of (4.29) converge to the integrands of (4.31) and of (4.30) respectively, and that there is no problem with commuting integration with taking the $\epsilon \to 0$ limit. The cases when X_1 and X_2 are not far away from X_3 can be treated in the same way as in the previous proof.

$\square$

Proof of (4.20) It will be convenient here to replace ϵ by 2ϵ and then take the $\epsilon \to 0$ limit. In all of the relevant diagrams two of the s's are constrained to be exactly 2ϵ apart and the third to be farther then 2ϵ from any of the previous two. It is harmless to assume that $s_2 = -\epsilon$, $s_3 = \epsilon$, $X(0) = 0$, and $s_1 = \tau$ with $|\tau| > 3\epsilon$. We will denote the ratio ϵ/τ by η.

With these notations one can see that the integrands corresponding to our diagrams can be written in pairs as follows: (ignoring the overall coefficient $-1/16\pi$)

$$
\begin{aligned}
E1 - E2 &= \sum_{\beta=\pm} 6_{ijki'j'k'}\dot{X}_\tau^{i'}\omega_{-\beta\eta\tau}^{j'}\dot{X}_{\beta\eta\tau}^{k'}T^{ijk}(X_\tau,X_{-\beta\eta\tau},X_{\beta\eta\tau})\\
-E6 - E9 &= \sum_{\beta=\pm} \epsilon_{mni}\epsilon_{ljk}\dot{X}_\tau^m\omega_\tau^n\dot{X}_{\beta\eta\tau}^l T^{ijk}(X_\tau,X_{-\beta\eta\tau},X_{\beta\eta\tau})\\
E7 + E8 &= \sum_{\beta=\pm} \epsilon_{mnj}\epsilon_{lki}\dot{X}_{-\beta\eta\tau}^m\omega_{-\beta\eta\tau}^n\dot{X}_\tau^l T^{ijk}(X_\tau,X_{-\beta\eta\tau},X_{\beta\eta\tau}).
\end{aligned}
$$

Remembering (4.5), (4.12), (4.14), and lemma 5.1 we see that in considering the $\epsilon \to 0$ limit we just need to show that

$$
\begin{aligned}
\lim_{\epsilon\to 0}\int_{|\tau|>T}\frac{d\tau}{\eta^a\tau^b}\sum_{\beta=\pm}\Big(6_{ijki'j'k'}\dot{X}_\tau^{i'}\omega_{-\beta\eta\tau}^{j'}\dot{X}_{\beta\eta\tau}^{k'} &+ \epsilon_{mni}\epsilon_{ljk}\dot{X}_\tau^m\omega_\tau^n\dot{X}_{\beta\eta\tau}^l\\
&+\epsilon_{mnj}\epsilon_{lki}\dot{X}_{-\beta\eta\tau}^m\omega_{-\beta\eta\tau}^n\dot{X}_\tau^l\Big)S_q^{p,ijk} = 0. \qquad (4.32)
\end{aligned}
$$

and that

$$\lim_{\epsilon\to 0}\int_{3\epsilon<|\tau|<T}\frac{\text{(same)}}{\eta^a\tau^b}d\tau = O(T) \qquad (4.33)$$

where T is some fixed small positive number and a and b are the exponents of η and τ as in equations (4.6), (4.7), and (4.8).

As in (4.32) τ is bounded from below we can use $\epsilon = \eta\tau$ to replace the limit there by an $\eta \to 0$ limit and then all that is required is to show that the summand there is $\sim \eta^{a+1}$. The relevant algebra will be carried out in the appendix using a computer.

The integration domain in (4.33) is symmetric and therefore we can replace the integration there with an integration over $3\epsilon < \tau < T$, replacing the integrand with

$$\sum_{\substack{\beta=\pm\\ \alpha=\pm}} \Big(6_{ijki'j'k'} \dot{X}^{i'}_{\alpha\tau} \omega^{j'}_{-\beta\eta\tau} \dot{X}^{k'}_{\beta\eta\tau} + \epsilon_{mni}\epsilon_{ljk} \dot{X}^{m}_{\alpha\tau} \omega^{n}_{\alpha\tau} \dot{X}^{l}_{\beta\eta\tau} + \epsilon_{mnj}\epsilon_{lki} \dot{X}^{m}_{-\beta\eta\tau} \omega^{n}_{-\beta\eta\tau} \dot{X}^{l}_{\alpha\tau} \Big) S^{ijk}_{1,2}\Big|_{\tau\to\alpha\tau}. \tag{4.34}$$

Simply integrating over τ now shows that to conclude the invariance proof we just need to show that (4.34) $= O(\eta^a\tau^b)$. Again, the relevant algebra will be carried out in the appendix using a computer.

□

Conclusion of the invariance proof What we've shown so far is that

$$\lim_{\epsilon\to 0} \delta\tilde{\mathcal{W}}_{2,\epsilon} = 0 \tag{4.35}$$

but what we need is

$$\delta\left(\lim_{\epsilon\to 0} \tilde{\mathcal{W}}_{2,\epsilon}\right) = 0.$$

Namely, we need to know that we can "commute" the $\epsilon \to 0$ limit with taking the variation $\delta/\delta\omega$. This follows from the following fact:

Fact If $X^{(t)} :\to \mathbf{R}^3;\ t \in [-1,1]$ is a smooth family of parametrized knots, then the convergences in (4.17) and in (4.35) are uniform in t.

To prove this fact simply observe that all the estimates in section 4.2 and in this section were, in fact, uniform for families of parametrized knots having a uniform upper bound on their first, second and third derivatives, a uniform lower bound on their first derivative, and a uniform lower bound on their distance from "self-intersecting".

□

4.4 Identifying $\tilde{\mathcal{W}}_2$

The last assertion of theorem 2 is that the invariant $\tilde{\mathcal{W}}_2$ that we have produced is essentially the second non-zero coefficient in the Conway polynomial of X. The Conway polynomial is defined by its behavior under flipping a crossing in a planar projection, so we will try to understand how $\tilde{\mathcal{W}}_2$ changes under such a flip.

Very briefly, it is clear that the difference in the value of $\tilde{\mathcal{W}}_2$ before and after a flip comes from a singularity in either of V_{ijk} or V_{ij} at the point where the flip occurs. Using the invariance that we have just proven one can 'straighten' the knot near a

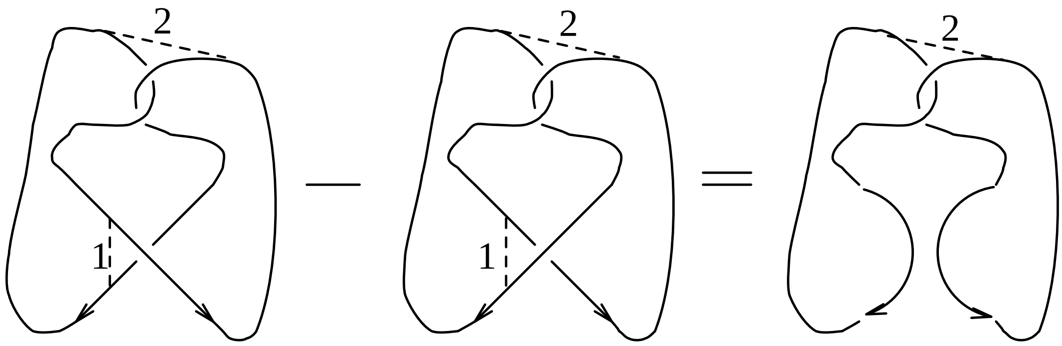


Figure 4.5. The change in $\tilde{\mathcal{W}}_2$ under a flip.

crossing point before flipping, and then it is easy to check in this case V_{ijk} contracted with the tangents of the knot in fact vanishes near the crossing point except if one of its arguments is on the upper branch of the crossing and the other is on the lower. V_{ijk} is then inversely proportional to the distance between its two arguments, and the fact that $1/r$ is integrable on $\mathbf{R}^2$ shows that this singularity can be neglected. Similarly considering diagram D one finds that the only singularity that remains is the one that occurs when the two arguments of the *same* propagator are arranged as propagator 1 in figure 4.5, and the other propagator can then be assumed to be away from the crossing. Repeating (3.4) for propagator 1 and then integrating over the location of the other propagator, marked 2 in the figure, it is clear that effectively we are calculating the linking number of the two knots that are created if the original knot is cut at the crossing as in the figure. It is easy to check from the definitions (see [31]) that this is exactly the same relation as the one that is satisfied by the second non-zero coefficient in the Conway polynomial of X, and so they coincide up to a constant shift. This constant shift is given by $\tilde{\mathcal{W}}_2$(unknotted circle). By invariance we can just calculate $\tilde{\mathcal{W}}_2$(the unit circle in the XY plane) and an explicit calculation shows (see [25]) that

$$\tilde{\mathcal{W}}_2(\text{the unit circle in the } XY \text{ plane}) = -\frac{\pi^2}{6}.$$

This concludes the proof of theorem 2.

4.5 Appendix: Some algebra

We include here the short computer routine that verifies few assertions that were made in sections 4 and 4.3. First, the routine itself. It is written in *Mathematica*™ — a symbolic mathematics language. For more information about this language see [48].

```
X[mu_] := {X1[mu],X2[mu],X3[mu]} ; Xd[mu_] := D[X[nu],nu] /. nu -> mu
```

```
X1[0]=X2[0]=X3[0]=0 ; w[mu_] := {w1[mu], w2[mu], w3[mu]}
ser[expr_] := Series[#,{var,0,ord}]& /@ expr
Xdtau    = ser[Xd[a tau]]      ;  wtau    = ser[w[a tau]]
Xdeps    = ser[Xd[b eta tau]]  ;  weps    = ser[w[b eta tau]]
Xdnegeps = ser[Xd[-b eta tau]] ;  wnegeps = ser[w[-b eta tau]]

t = lambda1 X[a tau] + lambda2 X[-b eta tau] + lambda3 X[b eta tau]
z1 = X[a tau] - t ; z2 = X[-b eta tau] - t ; z3 = X[b eta tau] - t
delta = IdentityMatrix[3]
S=Table[ser[Which[
   var==eta,{(z1[[i]]delta[[j,k]]+z2[[j]]delta[[k,i]]+z3[[k]]delta[[i,j]])
         /. lambda1 -> c2 eta ,
      z1[[i]] (Expand[z2[[j]]z3[[k]]]
         /. {lambda1^2 -> c5 eta^3 , lambda1 -> c4 eta^2})/eta^2},
   var==tau,{(z1[[i]]delta[[j,k]]+z2[[j]]delta[[k,i]]
        +z3[[k]]delta[[i,j]])/tau, z1[[i]]z2[[j]]z3[[k]]/tau^3}]],
   {i,3},{j,3},{k,3}]

sign = (Signature /@ (perm = Permutations[{1,2,3}]))
eps[f_]:=Sum[sign[[p]]sign[[q]](f@@Join[perm[[p]],perm[[q]]]),{p,6},{q,6}]
six[f_]:=eps[f[#3,#1,#4,#6,#2,#5]&] + eps[f[#6,#1,#4,#2,#3,#5]&]

e[type_]  :=
    six[S[[#1,#2,#3,type]]Xdtau[[#4]]Xdnegeps[[#5]]Xdeps[[#6]]&] /. b->1
e12[type_]:=six[S[[#1,#2,#3,type]]Xdtau[[#4]]wnegeps[[#5]]Xdeps[[#6]]&]
e69[type_]:=eps[S[[#3,#5,#6,type]]wtau[[#1]]Xdtau[[#2]]Xdeps[[#4]]&]
e78[type_]:=eps[S[[#6,#3,#5,type]]Xdnegeps[[#1]]wnegeps[[#2]]Xdtau[[#4]]&]
de[type_] :=Sum[e12[type] + e69[type] + e78[type] , {b,-1,1,2}]
```

The first paragraph of the routine defines X, $\dot{X}$, ω, and their expansions with respect to the externally defined variable `var` to order `ord` at the points $\alpha\tau$, $-\epsilon = -\beta\eta\tau$, and $\epsilon = \beta\eta\tau$.

The second paragraph defines `S[[i,j,k,1 or 2]]` to be $S^{ijk}_{1 \text{ or } 2}$ expanded with respect to the relevant variable. `S` is defined *differently* for `var=eta` then for `var=tau` — if `var=eta` then (4.6) and (4.7) mean that in S_1 one can make the replacement `lambda1 -> c2 eta` while (4.8) means that in S_2 the replacement `{lambda1^2 -> c5 eta^3 , lambda1 -> c4 eta^2}` can be made. It is easy to see that after the latter replacement has been made the expansion for S_2 will begin at η^2, and this justifies dividing it by η^2 and expanding everything to an order two less than is mentioned in sections 4 and 4.3. If `var=tau` the expansions for `z1`, `z2`, and `z3` begin at τ, and thus the definitions `S[[i,j,k,1]]`$= S^{ijk}_1/\tau$ and `S[[i,j,k,2]]`$= S^{ijk}_2/\tau^3$. This allows us to expand `S[[i,j,k,1]]` (`S[[i,j,k,2]]`) to an order lower by one (three) than the order required for S^{ijk}_1 (S^{ijk}_2).

The third paragraph contains the routines that do the $\epsilon_{\dots}$ and the $6_{\dots\dots}$ contractions, and the last paragraph defines the relevant diagrams.

We now include a *Mathematica*™ session produced using the above routine, for which I have chosen the not very imaginative name “file”.

```
Mathematica (sun4) 1.2 (November 6, 1989) [With pre-loaded data]
by S. Wolfram, D. Grayson, R. Maeder, H. Cejtin,
   S. Omohundro, D. Ballman and J. Keiper
with I. Rivin and D. Withoff
Copyright 1988,1989 Wolfram Research Inc.

In[1]:= var=eta; ord=1; << file

In[2]:= {e[1] , e[2]} /. {a->1 , eta->0}

Out[2]= {0, 0}

In[3]:= {de[1] , de[2]} /. a->1

                2        2
Out[3]= {O[eta] , O[eta] }

In[4]:= var=tau; ord=1; << file

In[5]:= {Sum[e[1],{a,-1,1,2}] , Sum[e[2],{a,-1,1,2}]}

                2        2
Out[5]= {O[tau] , O[tau] }

In[6]:= var=tau; ord=2; << file

In[7]:= {Sum[de[1],{a,-1,1,2}] , Sum[de[2],{a,-1,1,2}]}

                3        3
Out[7]= {O[tau] , O[tau] }
```

`Out[2]` and `Out[5]` prove equations (4.13) and (4.15), while `Out[3]` and `Out[5]` prove the assertions at the end of the invariance proof in section 4.3. This concludes the proof of the main theorem of this paper.

Remark Obtaining these eight expansions takes few hours of CPU time on a 1989 workstation.

Chapter 5

The stationary phase approximation

The purpose of this chapter[1] is to compute and examine the consequences of the stationary phase approximation of section 1.2.2. In [42] Witten has calculated the stationary phase approximation for the Chern-Simons path integral, finding that the effective coupling constant is shifted by half the Casimir number[2] $c_2(\mathcal{G})$ of the adjoint representation of the underlying group relative to the bare coupling constant k. His calculation was restricted to compact simple gauge groups, and one of the purposes of this chapter is to examine the (somewhat different) case of non-compact simple groups. The results of this chapter were obtained jointly with E. Witten, and are all included (in a somewhat different format) in [7].

5.1 Introduction

Recall from section 1.3.3 that the quadratic part of the gauge fixed Chern-Simons Lagrangian is given by

$$k\,cs(A_0) + \frac{k}{4\pi}\int_{M^3} \mathfrak{tr}\left(A \wedge D^{A_0}A + 2\phi D_i^{A_0}A^i + 2\bar{c}D_i^{A_0}D^{A_0,i}c\right) \tag{5.1}$$

where D^{A_0} denotes covariant differentiation with respect to a background flat connection A_0. If the gauge group G is simple and compact, then the inner product

$$\langle\varphi_1,\varphi_2\rangle = -\int_{M^3}\sqrt{g}\,\mathfrak{tr}(\varphi_1\varphi_2) \tag{5.2}$$

[1]Actually, in the logical order of things, this chapter deserves to appear before chapters 3 and 4. However, due to its less complete and less rigorous nature I've decided to place it after those two rigorous sections.

[2]The Casimir number $c_2(R)$ of a representation R of a simple Lie algebra $\mathcal{G}$ relative to some pre-chosen invariant bilinear form $\mathfrak{tr}$ on $\mathcal{G}$ is the ratio $tr_R/\mathfrak{tr}$. Namely, $c_2(R)$ is the constant for which $tr_R R(\mathcal{G}_a)R(\mathcal{G}_b) = c_2(R)\mathfrak{tr}\mathcal{G}_a\mathcal{G}_b$ for every $\mathcal{G}_{a,b} \in \mathcal{G}$.

is positive definite[3], and we can rewrite (5.1) as

$$k\,cs(A_0) - \frac{k}{4\pi}\left\langle \begin{pmatrix} A \\ \phi \end{pmatrix}, L_-^{A_0} \begin{pmatrix} A \\ \phi \end{pmatrix} \right\rangle + \frac{k}{2\pi}\left\langle \bar{c}, \Delta^{A_0} c \right\rangle$$

where Δ^{A_0} is the covariant Laplacian and $L_-^{A_0}$ is defined as in section 3.1:

$$L_-^{A_0} = (D^{A_0}\star + \star D^{A_0})J \qquad ; \qquad \begin{array}{l} JA = A \\ J\phi = -\phi \end{array} .$$

repeating the same[4] analysis as in section 1.2.2, we thus find that to lowest order in $1/k$,

$$\mathcal{W}(M^3,k) \sim \sum_{\text{flat } A_0} \frac{\det \Delta^{A_0}}{\sqrt{\left|\det L_-^{A_0}\right|}} e^{-i\frac{\pi}{4}\operatorname{sign} L_-^{A_0}} e^{ik\,cs(A_0)}.$$

(Here we have ignored an A_0-independent infinite power of $4\pi k$).

The problem with the above formula is that as it stands, $\det \Delta^{A_0}$, $\det L_-^{A_0}$, and $\operatorname{sign} L_-^{A_0}$ are all meaningless due to the infinite dimensionality of the spaces involved. A way around this was found by Ray and Singer [37] — they show when L is a suitable operator, the sum

$$\zeta(L,s) = \sum_{\text{eigenvalues } \lambda \text{ of } L} \lambda^{-s}$$

converges for $\mathrm{Re}(s)$ large enough, that the resulting *ζ-function* has a meromorphic continuation on the entire s plane, and that it is analytic at $s = 0$. Finally, they *define*

$$\det L = e^{-\zeta'(L)} \stackrel{\text{def}}{=} e^{-\zeta'(L,0)}.$$

Clearly, this definition agrees with the usual definition of the determinant in the finite dimensional case.

Similarly, one can define (following Atiyah, Patodi, and Singer [5])

$$\eta(L,s) = \sum_{\text{eigenvalues } \lambda \text{ of } L} \lambda^{-s} \operatorname{sign} \lambda$$

for $\mathrm{Re}(s)$ large enough, analytically continue to $s = 0$, and set

$$\operatorname{sign} L = \eta(L) \stackrel{\text{def}}{=} \eta(L,0).$$

With these definitions, we can set

$$\mathcal{W}_0^{\textit{regularized}} = \sum_{\text{flat } A_0} \exp\left(\frac{1}{4}\zeta'\left(\left(L_-^{A_0}\right)^2\right) - \zeta'\left(\Delta^{A_0}\right)\right) \exp -i\frac{\pi}{4}\eta\left(L_-^{A_0}\right) \exp ik\,cs(A_0). \tag{5.3}$$

---

[3] Here we have restricted our choice of $\mathfrak{tr}$ a bit further. Not only do we require that it will be invariant, namely a multiple of the Killing form, but we also insist that it will be a *positive* multiple of the negative definite Killing form.

[4] But remembering that for Fermionic Gaussian integrals $\int d\bar{c}dc\, e^{\bar{c}Jc} \propto \det(J)$ as in (1.13).

In the process of defining $\mathcal{W}_0^{regularized}$ we were forced to introduce a metric on M^3, and it is now not clear that our definition is independent of the choice of that metric. Part of the answer was already given by Ray and Singer in [37] — they proved that the ratio of determinants

$$\exp\left(\frac{1}{4}\zeta'\left(\left(L_-^{A_0}\right)^2\right) - \zeta'\left(\Delta^{A_0}\right)\right)$$

is, in fact, metric independent[5].

The signature $\eta(L_-^{A_0})$ turns out to be trickier. We will see in the next section that for an arbitrary connection A (not necessarily flat), the variation of $\eta(L_-^A)$ with respect to A is given by

$$\delta\eta\left(L_-^A\right) = -\frac{c_2(\mathcal{G})}{\pi^2}\int_{M^3} \mathfrak{tr}\, \delta A \wedge F^{A_0}. \tag{5.4}$$

This implies[6]

$$\eta\left(L_-^{A_0}\right) - \eta\left(L_-^0\right) = -\frac{2c_2(\mathcal{G})}{\pi} cs(A_0),$$

where L_-^0 is the standard L_- operator $(d\star + \star d)J$ twisted by the zero connection. Therefore, the "second half" of (5.3) can be rewritten as

$$\exp i\frac{\pi}{4}\eta\left(L_-^{A_0}\right)\exp ik\, cs(A_0) = \exp -i\frac{\pi \dim G}{4}\eta(L_-)\exp i(k + c_2(\mathcal{G})/2)cs(A_0). \tag{5.5}$$

The shift $k \to k + c_2(\mathcal{G})/2$ in the above formula is exactly the famous "shift in k" of Chern-Simons theories.

We still have to analyze the *metric* dependance of $\mathcal{W}_0^{regularized}$ — namely, the metric dependance of $\eta(L_-)$. Here we can appeal again to the Atiyah-Patodi-Singer theorem, which, in this case, says that

$$\frac{\delta\eta(L_-)}{\delta g} = \frac{1}{12\pi^2}\int_{M^3} tr\frac{\delta\omega_g}{\delta g}\wedge R^g = \frac{\delta cs(\omega^g)}{6\pi\delta g}, \tag{5.6}$$

where R^g is the curvature of the Levi-Civita connection ω^g of g. The situation now is similar to that of section 3.4 — $\mathcal{W}_0^{regularized}$ is not invariant, but it can be 'corrected' to give an invariant

$$\mathcal{W}_0^{renormalized} \stackrel{\text{def}}{=} \mathcal{W}_0^{regularized} e^{i\frac{\dim G}{24}cs(\omega^g)}$$

at the cost of having to *frame* M^3 — to choose a homotopy class of trivializations of the tangent bundle of M^3 — so that $cs(\omega^g)$ can be defined unambiguously.

[5] They have also conjectured that that ratio is equal to the square root of the Reidemeister-Franz torsion of M^3 with coefficients in the representation of $\pi_1(M^3)$ determined by A_0. This conjecture was later proven by Cheeger [15] and Müller [32] independently.

[6] This result can be deduced directly from the Atiyah-Patodi-Singer theorem [5].

5.2 The variation of η in the compact case

As a warm-up for the more challenging case of a non-compact group, in this section we will prove formula (5.4). For simplicity we will perform all our computations on a flat $\mathbf{R}^3$. A more complete treatment can be found in [7].

The first step is to rewrite L_-^A:

$$L_-^A = \sigma_i \left(\frac{\partial}{\partial x^i} + A_i \right).$$

Here σ_1, σ_2, σ_3 are the matrices representing multiplication by the quaternions i, j, k respectively on the four dimensional real vector space V underlying the quaternions $\mathbf{H}$:

$$\sigma_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; ; \; \sigma_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \; ; \; \sigma_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

This differential operator acts on $V \otimes \mathcal{G}$-valued functions on $\mathbf{R}^3$. The $\{\sigma_i\}$'s satisfy the following commutation relations:

$$\{\sigma_i, \sigma_j\} = -2\delta_{ij} \tag{5.7}$$
$$[\sigma_i, \sigma_j] = 2\epsilon_{ijk}\sigma_k. \tag{5.8}$$

We will attempt to calculate $\eta(L_-^A)$ using a result derived in [1] and in [11], and reviewed in [7]:

Theorem 3 *The variation of the η-invariant $\eta(D)$ of a differential operator D acting on a three dimensional space is given by*

$$\delta[\eta(D)] = -\frac{2C_{-1/2}}{\sqrt{\pi}}$$

where the form $C_{-1/2}$ is related to the asymptotic expansion of the heat kernel of D^2 by[7]

$$Tr(\delta D \exp -tD^2) = \frac{C_{-3/2}}{t^{3/2}} + \frac{C_{-1}}{t} + \frac{C_{-1/2}}{t^{1/2}} + \cdots$$

If $D^2 = -(\Delta + F)$ and the operator F can be considered as 'small' relative to Δ, one can determine the coefficients $C_{-m/2}$ using

$$\left\langle x \left| e^{t(\Delta+F)} \right| y \right\rangle \sim \left\langle x \left| e^{t\Delta} \right| y \right\rangle + \int_0^t ds \left\langle x \left| e^{s\Delta} F e^{(t-s)\Delta} \right| y \right\rangle + \cdots$$

[7]The T of Tr in the formula below is capitalized to emphasize the infinite dimensionality of the space involved.

To apply theorem 3, we first need to calculate $(L_-^A)^2$:

$$(L_-^A)^2 = \sigma_i\sigma_j(\partial_i + A_i)(\partial_j + A_j) = \sigma_i\sigma_j\partial_i\partial_j + \sigma_i\sigma_j\partial_i \circ A_j + \sigma_i\sigma_j A_i\partial_j + \sigma_i\sigma_j A_i A_j$$

using Leibnitz' rule

$$= \sigma_i\sigma_j\partial_i\partial_j + \sigma_i\sigma_j(\partial_i A_j) + (\sigma_i\sigma_j + \sigma_j\sigma_i)A_i\partial_j + \sigma_i\sigma_j A_i A_j$$

And replacing each $\sigma_i\sigma_j$ by $\frac{1}{2}(\{\sigma_i,\sigma_j\} + [\sigma_i,\sigma_j]) = -\delta_{ij} + \epsilon_{ijk}\sigma_k$ we get

$$= -\Delta - (\partial_i A_i) + \epsilon_{ijk}\sigma_k(\partial_i A_j) - 2A_i\partial_i - A_iA_i + \epsilon_{ijk}\sigma_k A_i A_j.$$

Now, according to theorem 3 the variation of $\eta(L_-^A)$ under $L_-^A \to L_-^{A+\delta A}$, that is to say, under

$$\sigma_l\left(\frac{\partial}{\partial x^l} + A_l\right) \to \sigma_l\left(\frac{\partial}{\partial x^l} + A_l + \delta A_l\right)$$

is given by $-\frac{C_{-1/2}}{\sqrt{\pi}}$ where $C_{-1/2}$ is given by:

$$\frac{C_{-1/2}}{t^{1/2}} = \int tr\,\sigma_l\delta A_l \int_0^t ds\,\left\langle x \left| e^{s\Delta} F e^{(t-s)\Delta} \right| x \right\rangle,$$

and where F is given by:

$$F = (\partial_i A_i) - \epsilon_{ijk}\sigma_k(\partial_i A_j) + 2A_i\partial_i + A_iA_i - \epsilon_{ijk}\sigma_k A_i A_j.$$

There is now no need to calculate — it is clear that as

$$tr\,\sigma_l = 0 \qquad ; \qquad tr\,\sigma_l\sigma_k = -4\delta_{lk}$$

we will have

$$\frac{C_{-1/2}}{t^{1/2}} = \int 4tr\,\delta A_l\,(\epsilon_{ijl}(\partial_i A_j) + \epsilon_{ijl}A_iA_j) \int_0^t ds\,\left\langle x \left| e^{s\Delta} e^{(t-s)\Delta} \right| x \right\rangle$$

(The expressions $\partial_i A_j$ and A_iA_j can be assumed to be independent of x — it easy is to see that their possible dependence would have anyway lead to lower order contributions). Using now the convolution property of the heat kernel we find that

$$C_{-1/2} = \frac{1}{2\pi\sqrt{\pi}} \int tr\,\epsilon_{ijl}\delta A_l\,(\partial_i A_j + A_iA_j) = \frac{c_2(\mathcal{G})}{2\pi\sqrt{\pi}} \int \mathrm{tr}\,\epsilon_{ijl}\delta A_l\,(\partial_i A_j + A_iA_j)$$

proving formula (5.4).

Remark. It is clear from the above calculations that when we calculated $(L_-^A)^2$ we could have ignored every term that has no σ matrix in it — because those terms when multiplied by $\sigma_l\delta A_l$ end up having exactly one σ in them, and thus end up having zero trace. In fact, one of those terms, $A_i\partial_i$, gives a vanishing contribution to the end result for another reason as well. Let us try to calculate the contribution due to it:

$$\int tr\,\sigma_l\delta A_l \int_0^t ds\,\left\langle x \left| e^{s\Delta} A_i\partial_i e^{(t-s)\Delta} \right| x \right\rangle$$

Again the dependence of A in x can be ignored as it leads only to lower order contributions, and we see that we first have to evaluate

$$A_i \left\langle x \left| e^{s\Delta} \partial_i e^{(t-s)\Delta} \right| x \right\rangle .$$

We can now use the fact that the integral kernel for the solution of the heat equation is a symmetric function of x and y to replace the above expression with:

$$A_i \frac{\partial}{\partial y_i} \left\langle x \left| e^{s\Delta} e^{(t-s)\Delta} \right| y \right\rangle \bigg|_{y=x} .$$

Using the semigroup property of the heat kernel we get

$$A_i \frac{\partial}{\partial y_i} \left\langle x \left| e^{t\Delta} \right| y \right\rangle \bigg|_{y=x} = \frac{A_i}{(4\pi t)^{3/2}} \frac{\partial}{\partial y_i} \left\langle x \left| e^{-\frac{(x-y)^2}{4t}} \right| y \right\rangle \bigg|_{y=x} = 0.$$

Clearly, a similar calculation will show that even if F had any other terms which are first order differential operators those would have added no further contributions to $\delta\eta(L_-^A)$.

5.3 The variation of η in the non-compact case

If the gauge group G is simple but not compact, then the inner product (5.2) is not positive definite, and the analysis of (1.5) breaks down. The reason for that is that in section 1.2.2 the phase of the integral was determined by the signature of the quadratic form approximating the Lagrangian near a stationary point. This signature is equal to the signature of a linear operator representing this form using a *positive definite* inner product, but if the quadratic approximation is written using an operator and an *indefinite* inner product, then its signature is effected both by the indefiniteness of the operator and that of the inner product. However, this can be easily resolved — all that one has to do is to *pick* a positive definite inner product and to reexpress the quadratic part of the Lagrangian in terms of the new inner product.

Pick a maximal compact subgroup G^c of G, and a positive definite inner product on $\mathcal{G}$ invariant under the Adjoint action of G^c, such that if $\mathcal{G}$ is written as the direct sum of the Lie algebra $\mathcal{G}^c$ of G^c and its orthogonal complement $\mathcal{G}^n$, then the original bilinear form that we started with, $\mathfrak{tr}$, is given by the matrix

$$n \stackrel{\text{def}}{=} \begin{pmatrix} I^{cc} & 0 \\ 0 & -I^{nn} \end{pmatrix} .$$

(I^{cc} and I^{nn} are, of course, the identity matrices of $\mathrm{End}[\mathcal{G}^c]$ and $\mathrm{End}[\mathcal{G}^n]$, respectively). Also, it is more convenient to replace the original gauge condition $\frac{k}{2\pi} D_i A^i = 0$ by $\frac{k}{2\pi} n D_i A^i = 0$. With these choices made, the operator to consider is not the same $L_-^{A_0}$, but a slight variation of it $\tilde{L}^{A_0}$, which will presently be described. Let $\tilde{\sigma}_i \in \mathrm{End}[\mathcal{G} \otimes V]$ be given by:

$$\tilde{\sigma}_i \stackrel{\text{def}}{=} \begin{pmatrix} I^{cc} \otimes \sigma_i & 0 \\ 0 & I^{nn} \otimes \bar{\sigma}_i \end{pmatrix}. \tag{5.9}$$

Where $\bar{\sigma}_i \in \mathrm{End}[V]$ are given by multiplication by the *opposite* orientation quaternions:

$$\bar{\sigma}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \; ; \; \bar{\sigma}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \; ; \; \bar{\sigma}_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is useful to note that the $\tilde{\sigma}$'s satisfy the following commutation relations:

$$\{\tilde{\sigma}_i, \tilde{\sigma}_j\} = -2\delta_{ij} \tag{5.10}$$

$$[\tilde{\sigma}_i, \tilde{\sigma}_j] = 2n\epsilon_{ijk}\tilde{\sigma}_k \tag{5.11}$$

After all those preliminaries, we can finally write $\tilde{L}^{A_0}$:

$$\tilde{L}^{A_0} \stackrel{\text{def}}{=} \tilde{\sigma}_i \left(\frac{\partial}{\partial x^i} + A_i \right).$$

Similarly to the compact case, we start our calculation by calculating $(\tilde{L}^A)^2$. Remembering the remark at the end of the previous section, we find that

$$[(\tilde{L}^A)^2]_{\text{relevant}} = \tilde{\sigma}_i\tilde{\sigma}_j\partial_i\partial_j + \tilde{\sigma}_i\tilde{\sigma}_j(\partial_i A_j) + \tilde{\sigma}_i\tilde{\sigma}_j A_i^{\tilde{\sigma}_j} A_j$$

(here the superscript $\tilde{\sigma}_j$ denotes conjugation by $\tilde{\sigma}_j$ — $A_i^{\tilde{\sigma}_j} \stackrel{\text{def}}{=} \tilde{\sigma}_j^{-1} A_i \tilde{\sigma}_j$). Using $\tilde{\sigma}_i\tilde{\sigma}_j = \frac{1}{2}\left(\{\tilde{\sigma}_i, \tilde{\sigma}_j\} + [\tilde{\sigma}_i, \tilde{\sigma}_j]\right)$ and equation (5.10),(5.11), we see that up to irrelevant pieces, the last expression equals

$$-\Delta + n\epsilon_{ijk}\tilde{\sigma}_k(\partial_i A_j) + n\epsilon_{ijk}\tilde{\sigma}_k A_i^{\tilde{\sigma}_j} A_j.$$

Just as in the compact case treated in the previous section we find now that the variation of $\eta(\tilde{L}^A)$ under $\tilde{L}^A \to \tilde{L}^{A+\delta A}$, that is to say, under

$$\tilde{\sigma}_l \left(\frac{\partial}{\partial x^l} + A_l \right) \to \tilde{\sigma}_l \left(\frac{\partial}{\partial x^l} + A_l + \delta A_l \right)$$

is given by $-\frac{C_{-1/2}}{\sqrt{\pi}}$ where $C_{-1/2}$ is given by:

$$\frac{C_{-1/2}}{t^{1/2}} = \int tr\,\tilde{\sigma}_l \delta A_l \int_0^t ds \left\langle x \left| e^{s\Delta} F e^{(t-s)\Delta} \right| x \right\rangle$$

and where F is given by:

$$F = -n\epsilon_{ijk}\tilde{\sigma}_k \left((\partial_i A_j) + A_i^{\tilde{\sigma}_j} A_j \right).$$

Therefore, we find that

$$\sqrt{\pi}\delta\eta(\tilde{L}^A) = \int tr\,\epsilon_{ijl}\delta A_l^{\tilde{\sigma}_l} n\left(\partial_i A_j + A_i^{\tilde{\sigma}_j} A_j\right). \tag{5.12}$$

We will now check that the last result, eq. (5.12) can be easily interpreted to be to the variation of the Chern-Simons number of the projection onto the subspace of compact generators of the connection A, with a coefficient proportional to the difference of the Casimir numbers of the representations of G^c on $\mathcal{G}^c$ and on $\mathcal{G}^n$.

We first wish to understand matrices of the form $A_i^{\tilde{\sigma}_j}$. Decomposing

$$A_i = \begin{pmatrix} A_i^{cc} & A_i^{cn} \\ A_i^{nc} & A_i^{nn} \end{pmatrix}$$

according to the decomposition $\mathcal{G} = \mathcal{G}^c \oplus \mathcal{G}^n$, it is easy to check that the answer is:

$$A_i^{\tilde{\sigma}_j} = \begin{pmatrix} A_i^{cc} & \tau_j A_i^{cn} \\ \tau_j A_i^{nc} & A_i^{nn} \end{pmatrix}$$

where $\tau_j \stackrel{\text{def}}{=} -\bar{\sigma}_j\sigma_j = -\sigma_j\bar{\sigma}_j$. Notice that the matrices τ_j are always diagonal with two 1's and two -1's on the diagonal:

$$\tau_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \; ; \; \tau_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \; ; \; \tau_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now come to understanding $-\sqrt{\pi}\delta\eta(\tilde{L}^A)/2$, that is, to understanding

$$C_{-1/2} = \frac{1}{(4\pi)^{3/2}}\int tr\,\epsilon_{ijl}(\delta_1 + \delta_2) \stackrel{\text{def}}{=} \frac{1}{(4\pi)^{3/2}}\int tr\,\epsilon_{ijl}\delta A_l^{\tilde{\sigma}_l} n\left(\partial_i A_j + A_i^{\tilde{\sigma}_j} A_j\right).$$

Writting

$$\delta A_l^{\tilde{\sigma}_l} = \begin{pmatrix} \delta A_l^{cc} & \tau_l \delta A_l^{cn} \\ \tau_l \delta A_l^{nc} & \delta A_l^{nn} \end{pmatrix}$$

we see that

$$\delta_1 = \begin{pmatrix} \delta A_l^{cc}\partial_i A_j^{cc} - \tau_l\delta A_l^{cn}\partial_i A_j^{nc} & * \\ * & \tau_l\delta A_l^{nc}\partial_i A_j^{cn} - \delta A_l^{nn}\partial_i A_j^{nn} \end{pmatrix}.$$

But the traces of the matrices τ_l vanishes, and so

$$tr\,\epsilon_{ijl}\delta_1 = 4tr\,\epsilon_{ijl}\left(\delta A_l^{cc}\partial_i A_j^{cc} - \delta A_l^{nn}\partial_i A_j^{nn}\right). \tag{5.13}$$

Similarly, we perform matrix multiplication and find that (for the same reason as before we can ignore terms in which a matrix $\tau_{l \text{ or } j}$ appears. In fact, we can even ignore terms in which a product $\tau_l\tau_j$ appears - this is because the anti-symmetrization

ϵ_{ijl} constrains l and j to be different, and it is trivial to verify that for *different* l and j one has $tr\,\tau_l\tau_j = 0$.)

$$tr\,\epsilon_{ijl}\delta_2 = tr\,\epsilon_{ijl}\begin{pmatrix} \delta A_l^{cc}A_i^{cc}A_j^{cc} & * \\ * & -\delta A_l^{nn}A_i^{nn}A_j^{nn} \end{pmatrix}$$

and so

$$tr\,\epsilon_{ijl}\delta_2 = 4tr\,\epsilon_{ijl}\left(\delta A_l^{cc}A_i^{cc}A_j^{cc} - \delta A_l^{nn}A_i^{nn}A_j^{nn}\right) \tag{5.14}$$

Equations (5.13) and (5.14) together show that $C_{-1/2}$ is in fact the variation of the Chern-Simons number of the projection a of the connection A onto the subspace of compact generators, with a coefficient proportional to the difference of the Casimir numbers of the representations of G^c on $\mathcal{G}^c$ and on $\mathcal{G}^n$:

$$\frac{\delta\eta(\tilde{L}^A)}{\delta A} = -\frac{1}{\pi^2}\int \epsilon^{ijl}\left(tr_{\mathcal{G}^c}\delta A_l^{cc}(\partial_i A_j^{cc} + A_i^{cc}A_j^{cc}) - tr_{\mathcal{G}^n}\delta A_l^{nn}(\partial_i A_j^{nn} + A_i^{nn}A_j^{nn})\right)$$

$$= -\frac{c_2(\mathcal{G}^c) - c_2(\mathcal{G}^n)}{\pi^2}\int \mathrm{tr}\,\epsilon^{ijl}\delta a^l(\partial_i a_j + a_i a_j) = -2\frac{c_2(\mathcal{G}^c) - c_2(\mathcal{G}^n)}{\pi^2}\frac{\delta cs(a)}{\delta a}. \tag{5.15}$$

If one ignores the difference between A and a, the above result means that in the case of a non-compact gauge group the effective value of k is shifted by $(c_2(\mathcal{G}^c) - c_2(\mathcal{G}^n))/2$ similarly to the shift $k \to k + c_2(\mathcal{G})/2$ observed in the compact case in (5.5).

The difference between A and a is a bit disturbing, however. The projection $P : A \to a$ depends on a *choice* of a *non-ad-invariant* positive definite metric on $\mathcal{G}$ and is not gauge covariant, making the result (5.15) not gauge invariant. This is a similar situation to the one encountered in (5.6) where the metric independece was broken by the regularization and the difficulty can be solved in a similar way — by adding to the original Lagrangian a *local* counter-term $\Delta\mathcal{L}$ that depends only on A, g and the pointwise choice of the projection P. The required counter-term is

$$\Delta\mathcal{L} = \frac{c_2(\mathcal{G}^c) - c_2(\mathcal{G}^n)}{32\pi c_2(\mathcal{G})}\int tr_{\mathcal{G}}[D^A T, T] \wedge F^A$$

where D^A is the covariant exterior derivetive twisted by A, F^A is the curvature of A, and $T = P - P^{\perp}$. Indeed one has

$$\frac{c_2(\mathcal{G}^c) - c_2(\mathcal{G}^n)}{2}cs(a) + \Delta\mathcal{L} = \frac{c_2(\mathcal{G}^c) - c_2(\mathcal{G}^n)}{2}cs(A)$$

correcting the non-gauge-invariance of (5.15).

Chapter 6

Some non-perturbative results

In [42] Witten has shown that the computation of (1.2) can be reduced to a problem in conformal field theory which can be solved giving a non-perturbative definition for the infinite dimensional integral (1.2). Before going into our perturbative analysis, let as first review his non-perturbative results.

Witten's definition is quite successful in that he can show how to use it to evaluate (1.2) precisely for *every* three manifold M^3 and link $\mathcal{X}$ in it, and not just calculate its leading large k asymptotics for $\mathbf{R}^3$, but it is less elementary and very particular to the Chern-Simons theory. There doesn't seem to be any direct relation between his way of calculating and the perturbative calculation shown here, and it is interesting to compare the two view points. Let us start by reviewing his results for a link in $\mathbf{R}^3$, as presented in [43]. As is shown there, $\mathcal{W}(\mathbf{R}^3, \mathcal{X}, k)$ considered as a function of k and the gauge group $G = SU(N)$ is in fact up to a simple change of variable the HOMFLY [23] polynomial of the link $\mathcal{X}$, which itself is a generalization of the Jones polynomial of $\mathcal{X}$.

Witten shows that to define $\mathcal{W}(\mathbf{R}^3, \mathcal{X}, k)$ unambiguously one needs to consider *framed* links instead of just links. That is to say, each component X_γ of the link has to be accompanied with a prescribed 'framing' — a choice up to homotopy of a nowhere vanishing section F_γ of the normal bundle of X_γ, or, more geometrically, a choice of a 'shadow' for each component as in the figure 6.1.

According to Witten, if the framing of link changes by a single twist, $\mathcal{W}$ get multiplied by $e^{2\pi i h}$, where h is a real number determined by k and the representation R_γ corresponding to the component of the link on which the twist was made. This is shown schematically in figure 6.2.

In the case where the underlying group G is $SU(N)$ for some positive integer N, and all the representations R_γ are just the defining representation of $SU(N)$ in $\mathbf{C}^N$, h is given by:

$$h = \frac{N^2 - 1}{2N(N + k)} \tag{6.1}$$

The difference between any two framings of a *single* knot is measured using a single integer — the number of signed twists required to change one framing to the

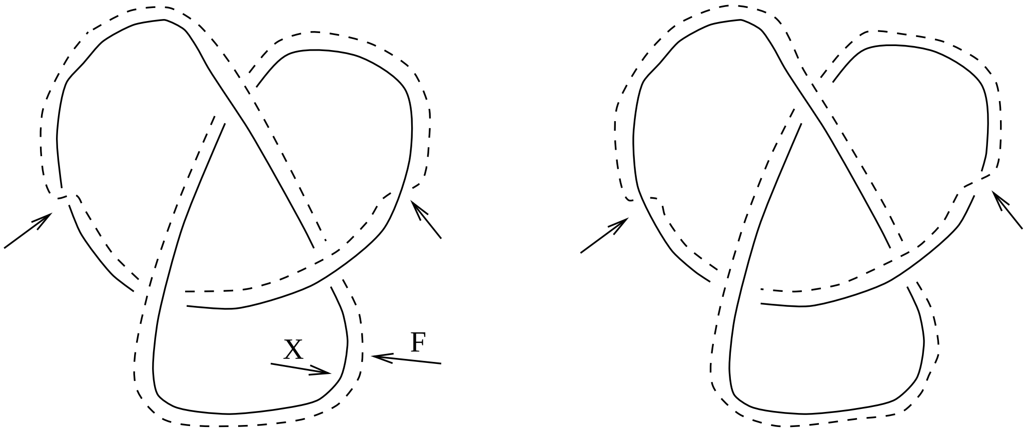


Figure 6.1. A knot with two of its possible framings. (The arrows indicate the differences between the two framings)

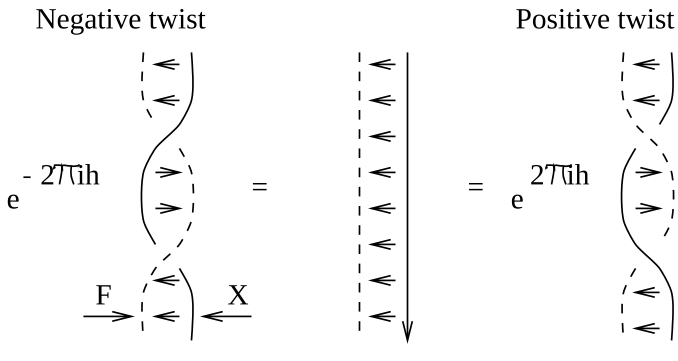


Figure 6.2. The change in $\mathcal{W}$ under a single twist.

other, and the above relation shows that for a link with several components we can in fact consider two framings to be equivalent if the total number of twists required to switch from one framing to the other is zero, counting all twists on all the components of the given link. With this identification for each link $\mathcal{X} = \{X_\gamma\}$ in $\mathbf{R}^3$ there is a unique preferred framing — the framing $\{F_\gamma\}$ for which the total linking number of $\mathcal{X}$ is 0:

$$\pounds(\mathcal{X}) \stackrel{\text{def}}{=} \sum_{\gamma_1,\gamma_2} \pounds(X_{\gamma_1}, F_{\gamma_2}) = 0$$

In this framing, Witten has shown that $\mathcal{W}(\mathbf{R}^3, \mathcal{X}, k)$ has the following three properties which allows one to calculate it for any given link:

1. For
$$q = e^{\frac{2\pi i}{N+k}} \tag{6.2}$$
one has
$$\mathcal{W}(\text{unknotted circle in } \mathbf{R}^3, k) = \frac{q^{N/2} - q^{-N/2}}{q^{1/2} - q^{-1/2}} \tag{6.3}$$
(In fact, this relation can be derived from the following two by using the third relation on the unknot whose planar projection is ∞)

2. If the link $\mathcal{X}$ is the *unlinked* union of $\mathcal{X}_1$ and $\mathcal{X}_2$ then
$$\mathcal{W}(\mathbf{R}^3, \mathcal{X}, k) = \mathcal{W}(\mathbf{R}^3, \mathcal{X}_1, k)\mathcal{W}(\mathbf{R}^3, \mathcal{X}_2, k) \tag{6.4}$$

3. Most important — the so called "skein relation" — if the three links L_0, L_+, and L_- differ only inside a small ball where they look as in figure 6.3,

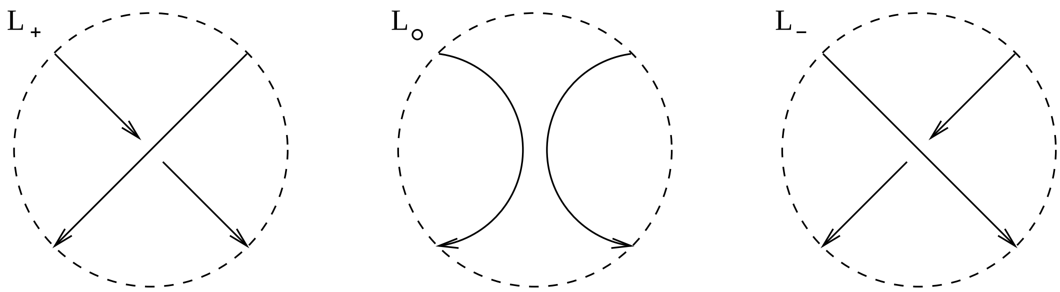


Figure 6.3. The links involved in the skein relation.

then the following relation holds:
$$-q^{N/2}L_+ + (q^{1/2} - q^{-1/2})L_0 + q^{-N/2}L_- = 0 \tag{6.5}$$
where for brevity we wrote $L.$ for $\mathcal{W}(\mathbf{R}^3, L., k)$.

To compare these results with ours we first need to expand them in powers of $1/k$, and thus we will write for a link $L.$

$$\mathcal{W}(\mathbf{R}^3, L., k) \sim N^{c.} + \frac{a.}{k} + \frac{b.}{k^2}.$$

From (6.3) and (6.4) it is clear that $c.$ is just the number of components of the link $L.$ if $L.$ is the unlinked union of unknotted circles. In addition, the zeroth order part of (6.5) reads $-N^{c_+} + 0 + N^{c_-} = 0$ and as L_+ and L_- always have the same number of components it means that the number of components of an arbitrary $L.$ is given by $c.$. The terms of orders $1/k$ and $1/k^2$ in (6.5) give the following two relations:

$$a_+ - a_- = 2\pi i(NN^{c_\pm} - N^{c_0}), \tag{6.6}$$

$$b_+ - b_- = 2\pi i a_0 + 2\pi i N(NN^{c_\pm} - N^{c_0}) - \pi i N(a_+ + a_-). \tag{6.7}$$

If L^{tw} is the same one component link as L, only with its framing twisted positively once, expanding the relation in figure 6.2 in powers of $1/k$ gives two additional relations:

$$a = a^{tw} + \pi i(N^2 - 1) \tag{6.8}$$

$$b = b^{tw} + \pi i a^{tw}\frac{N^2-1}{N} - \pi\frac{N^2-1}{N}\left(iN^2 + \frac{\pi}{2}(N^2-1)\right). \tag{6.9}$$

Theorem 4 *The following assertions hold for links in* $\mathbf{R}^3$*:*

1. *For a two component link* L_+*,* $\frac{1}{N(N^2-1)}a_+$ *is* $2\pi i$ *times the linking number of its two components.*

2. *For a single component knot* L_+ *not necessarily with its preferred framing,* $\frac{a_+}{(N^2-1)}$ *is* πi *times its self linking number.*

3. *For a single component knot* L *not necessarily with its preferred framing,* $\tilde{b} \stackrel{\text{def}}{=} \frac{1}{N(N^2-1)}Re\left(b - \frac{a^2}{2N}\right)$ *is framing independent, and is in fact equal to our* $\tilde{\mathcal{W}}_2(L)$*.*

All of these assertions are easy consequences of (6.6)-(6.9). For example:
Proof of 3 To get the framing independence of $\tilde{b}$ just use (6.8) and (6.9) to express it in terms of a^{tw} and b^{tw}, and then notice that the resulting expression differs from that of $\tilde{b}^{tw}$ only by the real part of an imaginary number. To show that $\tilde{b}$ is equal to $\tilde{\mathcal{W}}_2(L)$ we just need to show that they satisfy the same skein relation. But for knots $L_\pm$ with their preferred framings $a_\pm = 0$ by 2, and therefore using (6.7) one gets

$$\tilde{b}_+ - \tilde{b}_- = \frac{1}{N(N^2-1)}\mathrm{Re}(b_+ - b_-) = \frac{2\pi i}{N(N^2-1)}a_0$$

which by 1 equals to $-4\pi^2$ times the linking number of the two knots obtained by cutting $L_\pm$ as in figure 4.5. It is easy to check that $\tilde{b}(\text{the unknot}) = -pi^2/6$. $\square$

Chapter 7

Translating BRST to Feynman diagrams

7.1 The BRST argument

To show that the Lagrangian that we obtained gives rise to a metric independent theory in spite of the explicit appearance of a metric in it, we will now introduce the 'BRST' operator Q of Becchi, Rouet, Stora, and Tyupin [8, 39] — the odd derivation acting on the space of all functionals of $A, \phi, \bar{c}, c$, defined by the following formula:

$$Q = \int_{M^3} \left(-(\partial_i c^a + f^a_{bc} A^b_i c^c) \frac{\delta}{\delta A^a_i} + \phi^a \frac{\delta}{\delta \bar{c}^a} + \frac{1}{2} f^a_{bc} c^b c^c \frac{\delta}{\delta c^a} \right). \tag{7.1}$$

Which is more conventionally written as:

$$\begin{aligned} QA_i &= -(\partial_i + \mathrm{ad}\ A_i)c, & (7.2)\\ Q\phi &= 0, & (7.3)\\ Q\bar{c} &= \phi, & (7.4)\\ Qc &= \frac{1}{2}[c,c] = \frac{1}{2}\mathcal{G}_a f^a_{bc} c^b c^c. & (7.5) \end{aligned}$$

In (7.2) the expression "ad A_i" stands for the operator defined by $(\mathrm{ad}\ A_i)c \stackrel{\mathrm{def}}{=} [A_i, c]$, in (7.5) and (7.1), f^a_{bc} are the structure constants of $\mathcal{G}$, $[\mathcal{G}_b, \mathcal{G}_c] = f^a_{bc}\mathcal{G}_a$, and $[c,c]$ doesn't vanish because of the anti-commutativity of c.

Lemma 7.1.1 $Q\mathcal{L}_{tot}(A, \phi, \bar{c}, c) = 0$.

Lemma 7.1.2 *There exists a functional Λ of A, ϕ, $\bar{c}$ and c (that depends on δg^{ij}) such that under $g^{ij} \to g^{ij} + \delta g^{ij}$,*

$$\delta\mathcal{L}_{tot} = Q\Lambda.$$

Lemma 7.1.3 *Q corresponds to a vector field of zero divergence.*

Lemma 7.1.4 $Q\mathcal{O} = 0$.

Let us first use the above four lemmas to prove that

$$\mathcal{W} = \int \mathcal{D}\varphi \; \mathcal{O} e^{i\mathcal{L}_{tot}}$$

is formally metric independent [45]. Indeed, under $g^{ij} \to g^{ij} + \delta g^{ij}$

$$\begin{aligned}
\delta\langle\mathcal{O}\rangle &= \delta \int \mathcal{D}\varphi \; \mathcal{O}(\varphi) e^{i\mathcal{L}_{tot}} \\
&= i \int \mathcal{D}\varphi \; \mathcal{O}(\varphi) e^{i\mathcal{L}_{tot}} \delta\mathcal{L}_{tot} \\
&= i \int \mathcal{D}\varphi \; Q\Big(\mathcal{O}(\varphi) e^{i\mathcal{L}_{tot}} \Lambda\Big). \qquad (7.6)
\end{aligned}$$

Here we used φ as a collective name for A, ϕ, $\bar{c}$ and c and in the last equality we made use of the first two lemmas. Now we just use the third lemma and the well-known fact that the integral of a derivative taken using a divergence-free vector field is always zero to conclude our proof.

Proof of lemma 7.1.1 This is just a simple calculation — one just applies Q to $\mathcal{L}_{tot}$ and sees it after some algebra. I will present this algebra here in a way that will be useful for our later purposes. First, let us decompose $\mathcal{L}_{tot}$ to a sum of it's 'free' part and it's 'interaction' part, and to a sum of it's bosonic part and it's fermionic part:

$$\mathcal{L}_{bos} = \mathcal{L}_{\text{bos,free}} + \mathcal{L}_{\text{bos,int}}$$

$$= \frac{k}{4\pi} \int_{M^3} \mathfrak{tr} \left(A \wedge dA + 2\phi \partial_i A^i \right) + \frac{k}{4\pi} \int_{M^3} \mathfrak{tr} \frac{2}{3} \left(A \wedge A \wedge A \right)$$

$$\mathcal{L}_{\text{ferm}} = \mathcal{L}_{\text{ferm,free}} + \mathcal{L}_{\text{ferm,int}} = \frac{k}{2\pi} \int_{M^3} \mathfrak{tr} \left(\bar{c} \partial_i \partial^i c \right) + \frac{k}{2\pi} \int_{M^3} \mathfrak{tr} \left(\bar{c} \partial_i [A^i, c] \right)$$

Let us now calculate the variation under Q of each of those parts:

$$\begin{aligned}
Q\mathcal{L}_{\text{bos,free}} &= -\frac{k}{2\pi} \int_{M^3} \mathfrak{tr}(dc + [A, c]) \wedge dA + \phi \partial_i (\partial^i + \text{ad } A^i) c \qquad (7.7) \\
Q\mathcal{L}_{\text{bos,int}} &= -\frac{k}{2\pi} \int_{M^3} \mathfrak{tr}(dc + [A, c]) \wedge A \wedge A \\
&= -\frac{k}{2\pi} \int_{M^3} \mathfrak{tr}\, dc \wedge A \wedge A \qquad (7.8) \\
Q\mathcal{L}_{\text{ferm,free}} &= \frac{k}{2\pi} \int_{M^3} \mathfrak{tr} \left(\phi \partial_i \partial^i c - \frac{1}{2} \bar{c} \partial_i \partial^i [c, c] \right) \qquad (7.9) \\
Q\mathcal{L}_{\text{ferm,int}} &= \frac{k}{2\pi} \int_{M^3} \mathfrak{tr} \left(\phi D_i [A^i, c] + \bar{c} \partial_i [\partial^i c + [A^i, c], c] - \frac{1}{2} \bar{c} D_i \left[A^i, [c, c] \right] \right) \qquad (7.10)
\end{aligned}$$

It is now easy to see that the first term of (7.7) cancels (7.8), that the second term of (7.7) cancels the sum of the first term of (7.9) and the first term of (7.10), that the second term in (7.9) cancels the second order part of the second term of (7.10), and that the remaining terms of 7.10 cancel.

$\square$

Proof of lemma 7.1.2 Suppose that $g^{ij} \to g^{ij} + \delta g^{ij}$. Then

$$\delta \mathcal{L}_{tot} = \frac{k}{2\pi} \int_{M^3} \sqrt{g}\delta g^{ij} T_{ij}$$

with

$$\begin{aligned} T_{ij} &= \mathfrak{tr}\Big((\partial_i \phi) A_j + (\partial_i \bar{c})(\partial_j + \text{ad } A_j) c \\ &\qquad - \frac{1}{2} g_{ij} \left((\partial_k \phi) g^{kl} A_l + (\partial_k \bar{c}) g^{kl} (\partial_l + \text{ad } A_l) c \right) \Big) \end{aligned}$$

and then $T_{ij} = Q\lambda_{ij}$ for

$$\lambda_{ij} = \mathfrak{tr}\left((\partial_i \bar{c}) A_j - \frac{1}{2} g_{ij} (\partial_k \bar{c}) g^{kl} A_l \right)$$

that is:

$$\delta \mathcal{L}_{tot} = Q \left(\frac{k}{2\pi} \int_{M^3} \sqrt{g}\delta g^{ij} \mathfrak{tr} \left((\partial_i \bar{c}) A_j - \frac{1}{2} g_{ij} (\partial_k \bar{c}) g^{kl} A_l \right) \right) \stackrel{\text{def}}{=} Q\Lambda.$$

$\square$

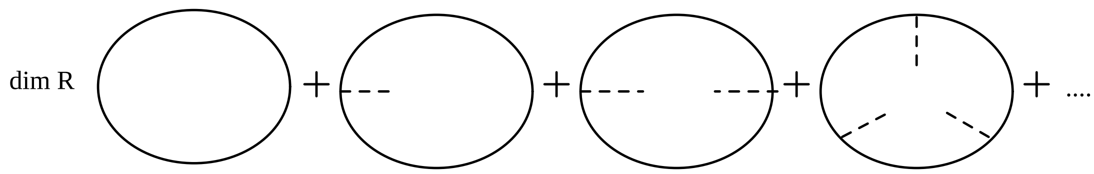
Proof of lemma 7.1.3

$$\begin{aligned} \text{div } Q &= \int_{M^3} \left(-\frac{\delta}{\delta A_i^a} (\partial_i c^a + f^a_{bc} A_i^b c^c) + \frac{\delta}{\delta \bar{c}^a} \phi^a + \frac{1}{2} \frac{\delta}{\delta c^a} f^a_{bc} c^b c^c \right) \\ &= \int_{M^3} (-f^a_{ac} c^c + 0 + f^a_{ac} c^c) = 0. \end{aligned}$$

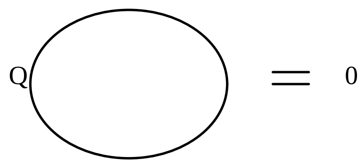
(Notice that for semisimple, Abelian and nilpotent Lie algebras each of the two terms above vanishes independently).

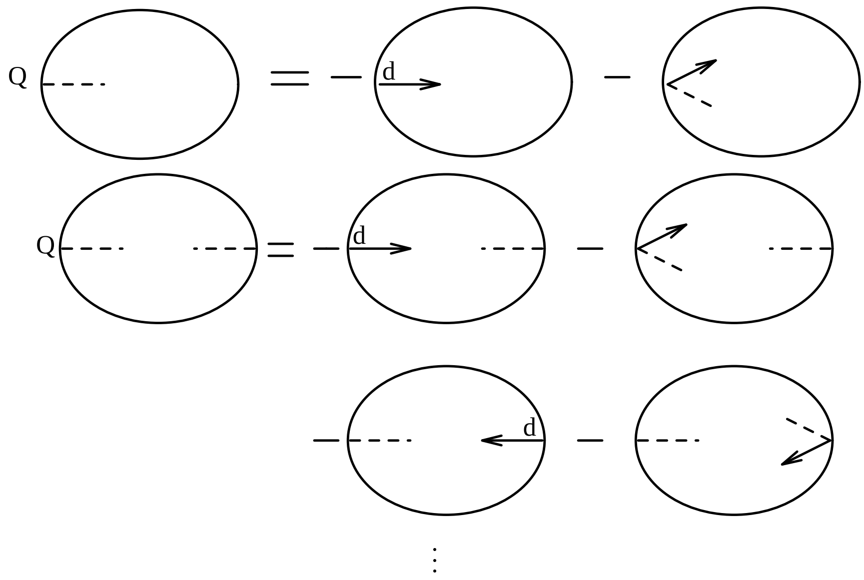
$\square$

Proof of lemma 7.1.4 This follows from the interpretation of $\mathcal{O}$ as the holonomy of A along $\mathcal{X}$, and the fact that the Q variation of A is just the infinitesimal gauge transformation corresponding to c. But for later reference, we can already write this proof in terms of diagrams. First, let us write the diagrams representing $\mathcal{O}$ itself:



Next, let us calculate $Q\mathcal{O}$ term by term:





Consider the terms that have an $\partial_i c$ vertex in them. There is, of course, integration over the position of this $\partial_i c$, and this is the integral of a gradient which can be replaced by a difference of boundary terms. These can be seen to be equal to the negatives of the terms that have an $[A_i, c]$ vertex.

$\square$

7.2 A simpler finite dimensional analogue

The invariance argument shown above is, of course, quite incomplete. It uses some familiar rules of integral calculus in an infinite dimensional setting in which the standard integration theory does not apply. However, what we have described in section 1.2 can be seen as being a *definition* of an integration theory in our infinite dimensional setting and we may wish to find how much of the standard rules of calculus still apply. The goal is to show that enough of standard calculus goes through, and that the invariance argument of the previous section can be translated into the well-posed language of Feynman diagrams. This will be done in the following two sections, beginning with a simpler finite dimensional example that highlights one of the key points.

In this section, we will show that for any $1 \leq q \leq N$ the perturbative expansion of

$$\int_{\mathbf{R}^N} d^N x \, \left(\partial_q P + ikP\lambda_{qj}x^j + 3ikP\lambda_{qjk}x^j x^k\right) e^{ik(\frac{1}{2}\lambda_{ij}x^i x^j + \lambda_{ijk}x^i x^j x^k)} \tag{7.11}$$

vanishes, where $\partial_q = \partial/\partial x^q$ and $P(x)$ is some monomial in x. Clearly, what we are now set to show is true — the integrand in the above integral is a derivative,

$$\partial_q \left(P(x)e^{ik(\frac{1}{2}\lambda_{ij}x^i x^j + \lambda_{ijk}x^i x^j x^k)}\right),$$

and if we believe the fundamental theorem of calculus, we are done. But in the infinite dimensional context that we really care about we don't have the fundamental theorem of calculus and therefore we would like to fine a direct combinatorial proof at the level of Feynman diagrams that (7.11) indeed vanishes.

Define

$$C \stackrel{\text{def}}{=} \begin{matrix}\text{Diagrammatic}\\ \text{expansion of}\end{matrix} \int_{\mathbf{R}^N} d^N x\, (\partial_q P) e^{ik(\frac{1}{2}\lambda_{ij}x^ix^j \lambda_{ijk}x^ix^jx^k)},$$

$$I \stackrel{\text{def}}{=} \begin{matrix}\text{Diagrammatic}\\ \text{expansion of}\end{matrix} \int_{\mathbf{R}^N} d^N x\, 3ikP\lambda_{qjk}x^jx^k e^{ik(\frac{1}{2}\lambda_{ij}x^ix^j+\lambda_{ijk}x^ix^jx^k)}.$$

and

$$F \stackrel{\text{def}}{=} \begin{matrix}\text{Diagrammatic}\\ \text{expansion of}\end{matrix} \int_{\mathbf{R}^N} d^N x\, ikP\lambda_{qj}x^j e^{ik(\frac{1}{2}\lambda_{ij}x^ix^j+\lambda_{ijk}x^ix^jx^k)},$$

It is clear that (7.11) is equal to $C + F + I$. We will show below that $F = -I - C$.

Following the rules of section 1.2.4, we see that the diagrams in F have the normal $\lambda_{ijk}x^ix^jx^k$ vertices and λ^{ij} propagators, and in addition to them two distinguished vertices. The first of these distinguished vertices corresponds to the monomial P (see figure 7.1), and the second (denoted by the 'magnet' symbol) corresponds to $i\lambda_{qj}x^j$ (see figure 7.2). Let us take a closer look at the second distinguished

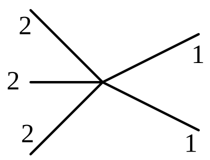


Figure 7.1. The vertex corresponding to the monomial $x_1^2x_2^3$.

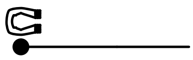
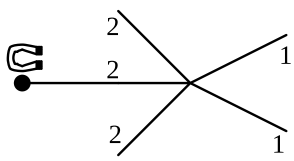
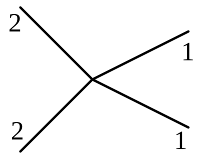


Figure 7.2. The vertex corresponding to $i\lambda_{qj}x^j$ has only one arc emanating from it because $i\lambda_{qj}x^j$ is of degree 1. The magnet points to the direction of 'attraction'.

vertex . When it appears in a diagram, say as in



the λ_{ij} in the vertex gets multiplied by its inverse — the propagator connecting to $*$ — and so the whole picture can be replaced by the vertex



which is one of the vertices corresponding to $-\partial_q *$! Remembering that  could have been connected to any of the other slots in $*$, we see that altogether all the ways to connect  to $*$ add up to give exactly the vertices that correspond to $-\partial_q *$. Now there are two possibilities for what $*$ could be. If $*$ is one of the regular $\lambda_{ijk}x^i x^j x^k$ vertices, then the process we just described (of 'pulling with the magnet' gives one of the diagrams in $-I$. If $*$ is the other distinguished vertex, the one corresponding to the monomial P, then 'pulling with the magnet' gives one of the diagrams in $-C$.

$\square$

7.3 Translating BRST to Feynman diagrams

Let us repeat the considerations of the previous section in the slightly more complicated case of the BRST invariance proof of section 7.1. Consider

$$F \stackrel{\text{def}}{=} \begin{array}{c}\text{Diagrammatic}\\ \text{expansion of}\end{array} \int \mathcal{D}\varphi\, (Q\mathcal{L}_{\text{free}})_{\text{untouched}}\, \mathcal{O}\Lambda e^{i\mathcal{L}_{tot}},$$

$$I \stackrel{\text{def}}{=} \begin{array}{c}\text{Diagrammatic}\\ \text{expansion of}\end{array} \int \mathcal{D}\varphi\, (Q\mathcal{L}_{\text{int}})_{\text{untouched}}\, \mathcal{O}\Lambda e^{i\mathcal{L}_{tot}},$$

and

$$C \stackrel{\text{def}}{=} \begin{array}{c}\text{Diagrammatic}\\ \text{expansion of}\end{array} \int \mathcal{D}\varphi\, (Q\Lambda)_{\text{untouched}}\, \mathcal{O}\Lambda e^{i\mathcal{L}_{tot}},$$

where the subscript "untouched" means that when calculating $Q\mathcal{L}_{\text{free}}$ and $Q\mathcal{L}_{\text{int}}$ no known identities are to be used to simplify the resulting expressions — they should just be left as they are.

We will see that:

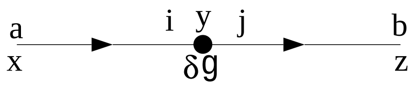
1. C is equal to the variation with respect to the metric of $\mathcal{W}$.
2. $F + I = 0$.
3. $F = -I - C$.

These assertions clearly imply $\frac{\delta}{\delta g}\mathcal{W} = 0$, which is what we've been aiming to prove.

Each of F, I, and C is a collection of diagrams made using the usual propagators and the usual X^2A, A^3, and $\bar{c}Ac$ vertices, only that each of those diagrams has an additional distinguished vertex of a form determined by the terms in $(Q\mathcal{L}_{\text{free}})_{\text{untouched}}$, $(Q\mathcal{L}_{\text{int}})_{\text{untouched}}$, and $Q\Lambda$. In addition, the diagrams in F and I will have a second distinguished vertex, corresponding to Λ. For example, as $Q\Lambda$ has in it a term:

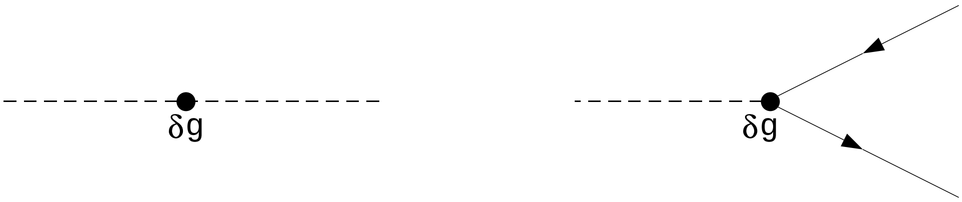
$$\frac{2}{4\pi}\int_{M^3}\sqrt{g}\,\delta g^{ij}\,\mathrm{tr}(\partial_i\bar{c})\partial_j c, \tag{7.12}$$

some of the diagrams in C will have in them a single distinguished vertex of the form



$$\longrightarrow \int_{M^3} \sqrt{g}\,\delta g^{ij} t_{ab} \left(\partial_i^y G^{db}(y,z)\right) \left(\partial_j^y G^{ac}(x,y)\right)$$

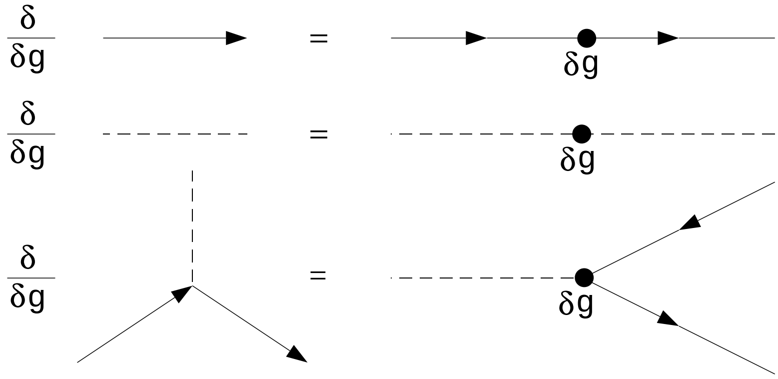
The other diagrams in C will have a distinguished vertex of either of the following forms:



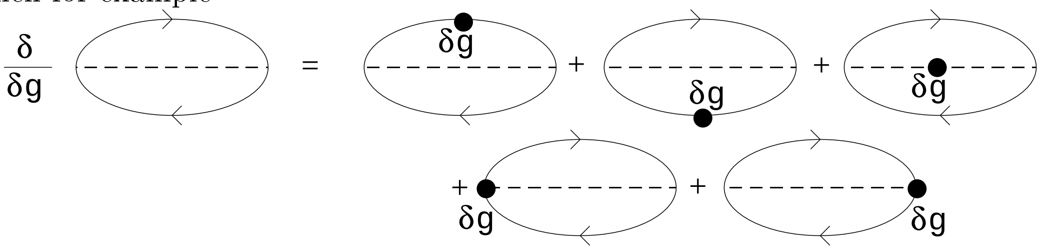
Proof of 1. Using

$$\delta\left(D^{-1}\right) = -D^{-1}(\delta D)D^{-1} \tag{7.13}$$

which holds for every linear operator D, one can see that



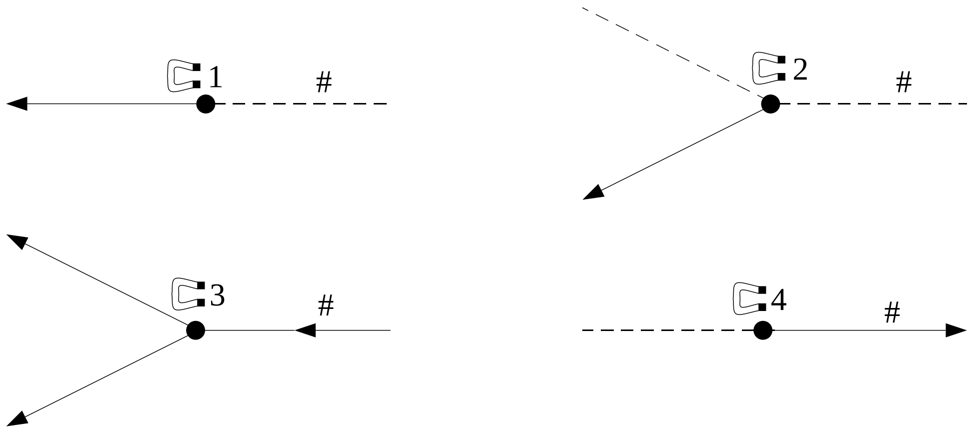
and then for example



These are exactly the diagrams in C! (And it turns out that the combinatorics works out right as well).

Proof of 2. Just remove the subscripts "untouched" and reread the proof of lemma 1.

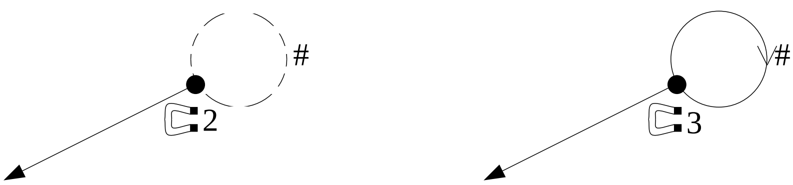
Proof of 3. Just as in the previous section, the diagrams in F will all have a distinguished vertex of one of the following four kinds, corresponding to the four terms in (7.7) and (7.9):



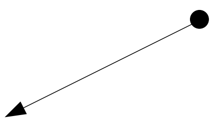
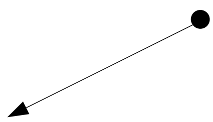
in each of those vertices, the slot marked by # has a differential operator acting on it. When a propagator is connected to one of those slots, the relations *defining* the propagator can be used to replace the propagator and the slot to which it is connected by a δ-function, effectively calculating the variation under Q of the vertex on the other end of that propagator.

There are now few possibilities as for where does that other end land.

1. The slot # on a vertex might be connected by a propagator to another slot on the same vertex . Here are the two such possibilities:

   When # is replaced by a δ-function as explained above, the resulting vertices are:

   
   and
   

   These two vertices are identical but with opposite signs, and therefore they cancel. This is exactly the fact proven in lemma 3 — that div $Q = 0$.

2. The distinguished vertex marked by a might be connected through the slot # to an X^2A vertex. After the connecting propagator is replaced by a δ-function as usual, we get exactly the diagrams in $Q\mathcal{O}$. These were shown to add up to zero in the proof of lemma 7.1.4.

3. The distinguished vertex might be connected through the slot # to an internal vertex of the diagram, of type A^3 or $\bar{c}Ac$. In this case the propagator

connecting the two vertices is replaced by a δ-function, the resulting diagram will have a distinguished vertex which appears in $-\left(Q\mathcal{L}_{\text{int}}\right)_{\text{untouched}}$, and so we get just the diagrams in $-I$.

4. The distinguished vertex  might be connected through the slot # to the other distinguished vertex - the one corresponding to Λ. In this case the propagator connecting the two vertices is replaced by a δ-function, the resulting diagram will have a single distinguished vertex, of the form $-Q\Lambda$. These are exactly the diagrams in $-C$.

$\square$

Chapter 8

The isotopy invariance argument

In this chapter we will prove (algebraicly, without the necessary analysis which is not yet done) that the perturbative coefficients $\mathcal{W}_m(X)$ are invariants of knots embedded in a flat $\mathbf{R}^3$. Of course, if $\mathcal{W}_m(X)$ is a topological invariant (does not depend on the metric g), then it has to be invariant under isotopies of the knot X, and so what we are set to show is actually a corrollary of the result of the previous chapter. However, the proof below differs in some ways from the proof in chapter 7, and this makes presenting this alternative proof worthwhile. The main advantage of the proof in this chapter is that it 'lives' entirely in flat space, and therefore it seems that it will be easier to supplement it with the necessary convergence analysis. Also, this proof is much more explicit, and makes the mechanism by which the variations of some diagrams cancel the variations of others much clearer.

8.1 Feynman rules in flat space

The Feynman rules in flat space are, of course, specializations of the rules given in chapter 2. However, in flat space[1] these rules can be generalized slightly. It turns out that the only way perturbation theory (in this case) depends on the Lie-algebra is through the numerical weights that are assigned to each diagram D by the contraction of all the Lie-algebra indices in $\mathcal{E}(D)$, and that the invariance proof below works even if these numerical weights are replaced by arbitrary weights, so long as these weights satisfy certain relations that will be described below. Other solutions of these relations (that do not necessarily come from a Lie-algebra) might exist, and such solutions might correspond to new link invariants.

We therefore redefine $\mathcal{W}_m(X)$ to be given by

$$\sum_{D\text{'s of order } m} \frac{C(D)}{S(D)} \int \mathcal{E}(D), \tag{8.1}$$

where $S(D)$ is defined just as in chapter 2, $\mathcal{E}(D)$ is defined as in chapter 2 only without including the Lie-algebra indices a, b, $\ldots$, and the $C(D)$'s are arbitrary

[1] Or actually, in arbitrary space but relative to the trivial background connection.

weights that ‘blind’ to the difference between gauge and ghost propagators and the difference between A^3 and $\bar{c}Ac$ vertices[2] and satisfy the following relations:

The “IHX” relation: Let the diagrams I, H, and X be identical outside a small domain, inside of which they look as in figure 8.1. Then their weights are expected to satisfy

$$C(I) = C(H) - C(X). \tag{8.2}$$

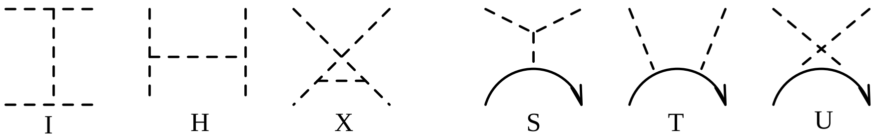


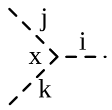
Figure 8.1. The diagrams I, H, and X, and the diagrams S, T, and U.

The “STU” relation: Let the diagrams S, T, and U be identical outside a small domain, inside of which they look as in figure 8.1. Then their weights are expected to satisfy

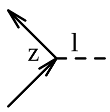
$$C(S) = C(T) - C(U). \tag{8.3}$$

Remark Actually, a little more care is necessary. The vertex A^3 as it was defined in (2.2) is symmetric with respect to the three propagators emanating from it, being a product of two anti-symmetric terms. In the A^3 vertex that we use in this chapter the tensor t_{abc} is removed, and so our A^3 vertex is *anti-symmetric*. Therefore, if we want to have unambiguous meaning to the Feynman rules, we must *choose* an orientation to each of the A^3 vertices in D — for each A^3 vertex, choose one of the two possible cyclic orderings of the three propagators meeting in that vertex. We assume that $C(D) = -C(D')$ if D' differs from D only in the orientation of a single vertex, and we use the convention that in a planar projection of a diagram each of the vertices is oriented *counterclockwise* ($\circlearrowleft$).

With our simplifying assumptions, some of the rules of chapter 2 become a bit simpler:



$$\longrightarrow \frac{i}{2\pi}\int_{M^3} dx\, \epsilon^{ijk}, \tag{8.4}$$



$$\longrightarrow \frac{1}{2\pi}\int_{M^3} dz\, \partial_z^l, \tag{8.5}$$

$$\overset{i}{\underset{x}{-}} - - \overset{j}{\underset{y}{-}} \longrightarrow V_{ij}(x,y) = \frac{i\epsilon^{ijk}(x-y)^k}{2|x-y|^3}, \tag{8.6}$$

and

$$\underset{x}{}\!\!\longrightarrow_{y} \longrightarrow G(x,y) = \frac{1}{2|x-y|}. \tag{8.7}$$

[2] Namely, if in a diagram D a loop of ghost ($\longrightarrow$) propagators connected by $\bar{c}Ac$ vertices is replaced by a loop of gauge ($- - - -$) propagators connected by A^3 vertices, then $C(D^{\text{before}}) = C(D^{\text{after}})$.

8.2 The variation of a diagram and the spider's journey

The m'th term $\mathcal{W}_m$ in the perturbative expansion of $\mathcal{W}(X,k)$ is given by a weighted sum of integrals of certain algebraic expressions which are most neatly represented by Feynman diagrams as in (8.1), (8.4)-(8.7). Our aim in the rest of this chapter is to prove[3] that under $X \to X + \delta X = X + \omega$,

$$\delta \mathcal{W}_m = \sum_D \frac{C(D)}{S(D)}\,\delta \int \mathcal{E}(D) = 0.$$

To do that, we have to calculate $\delta \int \mathcal{E}(D)$ for an arbitrary diagram D.

Let us first describe the 'main part' of the computation, disregarding various boundary and contact terms which will be the subject of the next section. Checking formulae (2.1) and (2.4) we see that the '$V_{ij}(x,y)$' connected to each X^2A vertex in D can be regarded as 1-form (with respect to either the variable x or the variable y), and that the X^2A vertex together with the s integration can be interpreted as the integral of that 1-form along the 1-cycle represented by a segment of the knot X. It is therefore clear that when the knot X is deformed, the variation of our integral $\int \mathcal{E}(D)$ (whose only X dependence is in the X^2A vertices) is given[4] by the evaluation of the exterior derivative of V on the infinitesimal surface S spanned by the deformation of X. This statement is reproduced in diagrams in figure 8.2. In that figure, a new

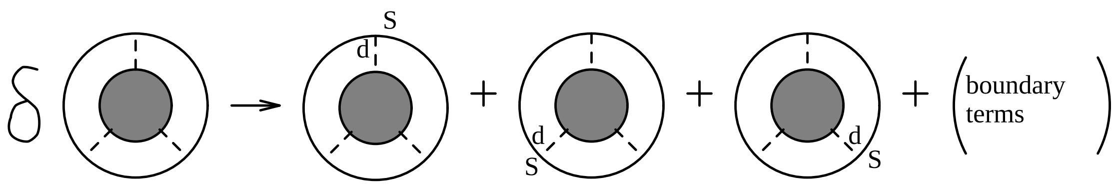
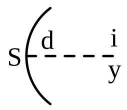


Figure 8.2. The six diagrams arising from the computation of $\delta \int \mathcal{E}(D)$ for D with 3 type X^2A vertices.

vertex is introduced, corresponding to the evaluation of $d^L V$ on S:



$$\longrightarrow -\dot{X}^j \omega^k \left(\frac{\partial}{\partial x^k} V_{ji}(x,y) - \frac{\partial}{\partial x^j} V_{ki}(x,y) \right)\bigg|_{x=X(s)}. \tag{8.8}$$

We see that in calculating $\delta \int \mathcal{E}(D)$ we find expressions that involve $d^L V$. Whenever such a term is encountered, we will use 'the key relation' of chapter 3.1 to replace it

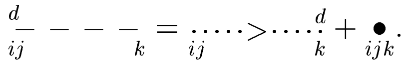
[3] *Formally prove.* Namely, present the *algebra* and *combinatorics* without considering the much harder *analysis* problems.

[4] Well, just almost given. There is a boundary correction which will be discussed in the next section.

by the right hand side of that relation. Recall that the key relation states that there exists a $(2,0)$-form F on $\mathbf{R}^3$ for which

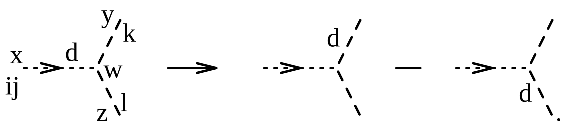
$$(d^L V)_{ij,k}(x,y) = (d^R F)_{ij,k}(x,y) + 2\pi i \epsilon_{ijk}\delta(x-y). \tag{8.9}$$

In diagrams, the relation (8.9) is reexpressed as



(8.10)

The last relation that we will use repeatedly is a combination of integration by parts and Leibnitz' rule described by the following diagram:



(8.11)

The corresponding formula is:

$$\frac{i}{2\pi}\int_{\mathbf{R}^3} dw\, \epsilon^{mnp}\left(\frac{\partial}{\partial w^m}F_{ij,-}(x,w)\right)V_{pk}(w,y)V_{nl}(w,z)$$

$$= \frac{-i}{2\pi}\int_{\mathbf{R}^3} dw\, \epsilon^{mnp}F_{ij,-}(x,w)\left(V_{nl}(w,z)\frac{\partial}{\partial w^m}V_{pk}(w,y) + V_{pk}(w,y)\frac{\partial}{\partial w^m}V_{nl}(w,z)\right)$$

$$= \frac{i}{4\pi}\int_{\mathbf{R}^3} dw\, \epsilon^{mnp}F_{ij,-}(x,w)\left(V_{nl}(w,z)(d^L V)_{pm,k}(w,y) - V_{pk}(w,y)(d^L V)_{mn,l}(w,z)\right).$$

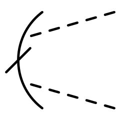
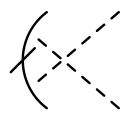
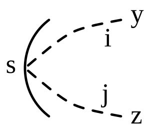
Summarizing, we first compute $\delta\int\mathcal{E}(D)$ as in figure 8.2, and then alternate replacing $d^L V$ by $d^R F$ as in (8.10) and integrating by parts as in (8.11). We can visualize this procedure by imagining a spider walking on our diagram on gauge $(----)$ propagators, beginning from some X^2A vertex, changing every gauge $(----)$ propagator that he had followed to a dotted $(\cdots\cdots>\cdots\cdots)$ propagator as in (8.10), and deciding whether to turn left or right whenever he reaches an A^3 vertex as in (8.11). The variation $\delta\int\mathcal{E}(D)$ is then given by a sum over all possible 'spider walks' on D of various boundary and contact terms that we have so far ignored and over all the 'deadends' — spider walks that cannot be continued further because the spider arrived at an X^2A vertex or a $\bar{c}Ac$ vertex, or has stepped on his own footsteps. We will consider all these boundary terms, contact terms, and deadends in the next section.

8.3 Boundary terms, contact terms, and deadends

8.3.1 The beginning of the journey

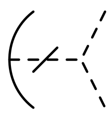
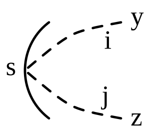
There are two types of diagrams produced in the evaluation of $\delta\int\mathcal{E}(D)$ even before the spider begins his journey. The first of them is the boundary term in figure 8.2 —

if the 1-form $V_{\cdot i}(\cdot, y)$ was evaluated on a closed cycle, there would have been no need for a correction in figure 8.2. But actually, it is evaluated on a cycle whose ends are given by two other X^2A vertices in D, and more care need to be taken near the ends. Stokes' theorem says that the integral of $V_{\cdot i}(\cdot, y)$ around the *complete* boundary of the part of S lying between these two X^2A vertices is given by (8.8). This boundary is made of four pieces — two long and almost parallel pieces that follow X and whose difference is exactly what we are trying to compute, and two infinitesimal pieces near the ends (see figure 8.3). The contributions to (8.8) from the two latter pieces needs to be subtracted off, and this is done by the following '$R1$' vertices:

The context:	The vertex $R1$:	The formula for $R1$:
 ; $-$ 		$\left(\dot{X}^k\omega^l - \dot{X}^l\omega^k\right) V_{li}(X,y)V_{kj}(X,z)$

The above rectangle is the form in which all the contributions to $\delta\mathcal{W}_m$ will be described. The left most column is the 'context column' that describes the context in which the presently discussed term appears — our term appears whenever there are two neighboring X^2A vertices in a diagram D, and we are considering one of them as the boundary of the other's domain of integration. The slash (/) on the knot segment connecting these two vertices indicates that the present contribution comes when the length of this segment *vanishes.* The center column is a diagram part that serves as the *symbol* of the currently discussed contribution to $\delta\int\mathcal{E}(D)$. To get the precise formula for this contribution, replace the symbol $R1$ by the formula in the right most column, and proceed to evaluate the other parts of D as in section 8.1.

The second contribution to $\delta\int\mathcal{E}(D)$ that arises even before the beginning of the spider's journey is the contact term arising from the δ-function in (8.10), when this formula is first applied:

The context:	The vertex $R2$:	The formula for $R2$:
		$\epsilon_{klm}\epsilon^{mnp}\omega^k\dot{X}^l V_{nj}(X,z)V_{pi}(X,y)$

8.3.2 The journey

During the journey itself, in which the operations (8.11) and (8.10) are alternated, there is only one kind of 'left over' contribution — the contact term arising from the δ-function in (8.10):

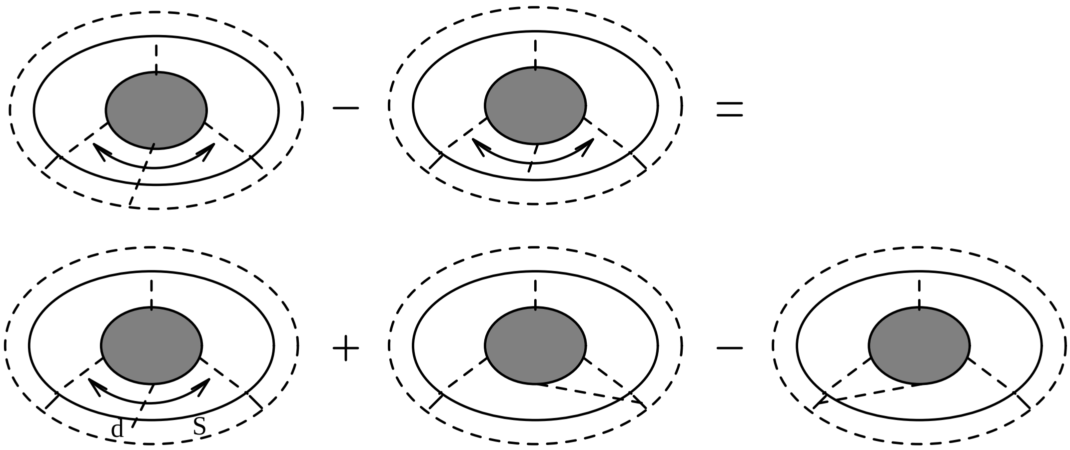
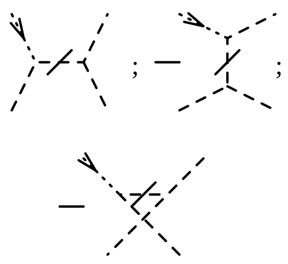
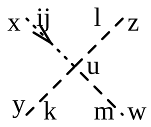
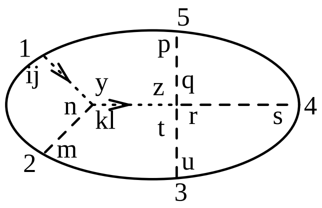


Figure 8.3. The boundary term $R1$: In these diagrams, the solid ellipses represent the knot X and the dashed ellipses represent the deformed knot $X+\omega$. The first two diagrams represent the part of the contribution to $\delta\int\mathcal{E}(D)$ coming from varying the position of *one* of the X^2A vertices in D. This X^2A vertex is integrated over a range (marked by a double arrow $\leftrightarrow$) bounded by two neighboring X^2A vertices. By Stokes' theorem, the quantity that we are interested in, the difference of the first two diagrams, is given by an integral of d^LV on the variation surface S (represented by the third diagram), plus the evaluation of V on the two short segments connecting the solid and the dashed ellipses near the bounding X^2A vertices. This last contribution is given by the term $R1$.

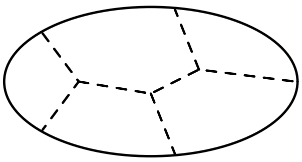
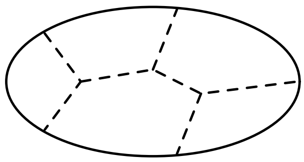
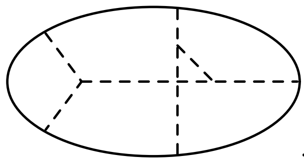
The context:	The vertex $R3$:	The formula for $R3$:
		$\begin{aligned} &\frac{-i}{4\pi}\int_{\mathbf{R}^3} du\, \epsilon^{npq}\epsilon_{pqr}\epsilon^{rst}F_{ij,-}(x,u) \\ &\cdot V_{kn}(y,u)V_{ms}(w,u)V_{lt}(z,u) \\ &= \frac{-i}{2\pi}\int_{\mathbf{R}^3} du\, \epsilon^{nst}F_{ij,-}(x,u) \\ &\cdot V_{kn}(y,u)V_{ms}(w,u)V_{lt}(z,u) \end{aligned}$

An example for a term of this sort will be the term



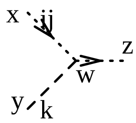
$$\longrightarrow \quad \frac{-1}{8\pi^2}\int_{\text{cyclicly ordered } s_{1-5}} ds_{1-5}\, dy\, dz\, \omega^i(s_1)\dot{X}_1^j\dot{X}_2^m\dot{X}_3^u\dot{X}_4^s\dot{X}_5^p \cdot F_{ij,-}(X_1,y)\epsilon^{nkl}V_{mn}(X_2,y)F_{kl,-}(y,z) \cdot \epsilon^{qrt}V_{uq}(X_3,z)V_{sr}(X_4,z)V_{pt}(X_5,z) \tag{8.12}$$

that arises in the variation of the diagrams

 ,  and  .

Notice that in the translation process in (8.12) we used the following two rules to deal with dotted ($\cdots\cdots{>}\cdots\cdots$) propagators and the F^2A vertex connecting two dotted and one gauge propagator, in addition to the standard rules of section 8.1:

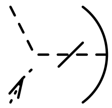
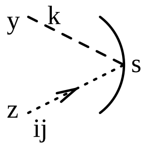
1.
$$\underset{ij}{\overset{x}{\cdot}}\cdots{>}\cdots\underset{-}{\overset{y}{\cdot}} = F_{ij,-}(x,y) \qquad \left(= \epsilon_{ijk}\frac{i(x-y)^k}{2|x-y|^3}\right). \tag{8.13}$$

2.


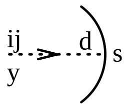
$$\longrightarrow \frac{i}{4\pi}\int_{\mathbf{R}^3} dw\, \epsilon^{lmn}F_{ij,-}(x,w)V_{kl}(y,w)F_{mn,-}(w,z) \tag{8.14}$$

8.3.3 The spider returns to the link

Right before the spider arrives at the link back again we get the following contact contribution, as usual from the δ-function in (8.10):

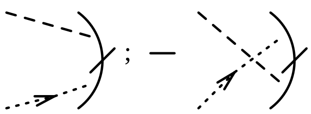
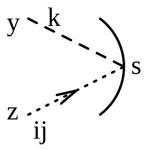
The context:	The vertex $R4$:	The formula for $R4$:
		$-\frac{1}{2}\epsilon^{lmn}\epsilon_{mnp}\dot{X}^p F_{ij,-}(z,X)V_{kl}(y,X)$ $= -\dot{X}^l F_{ij,-}(z,X)V_{kl}(y,X)$

When the spider arrives at the link, we get the following 'dead end' contribution:

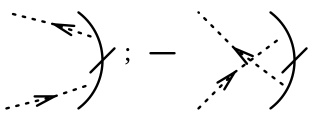
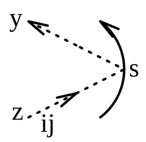


$$\longrightarrow -\dot{X}^k \frac{\partial}{\partial z^k} F_{ij,-}(y,z)\bigg|_{z=X} \tag{8.15}$$

Notice that here we are taking the line integral of a gradient $\left(\frac{\partial}{\partial z^k}F_{ij,-}(y,z)\right)$ along a segment of the knot X. Thus by the fundamental theorem of calculus (8.15) can be written as the difference of the values of $F_{ij,-}(y,X(s))$ at the two end points of the line of integration. Such an endpoint might be a regular X^2A vertex, in which case we get the term:

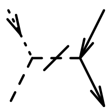
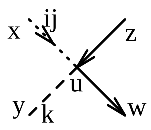
The context:	The vertex $R5$:	The formula for $R5$:
		$\dot{X}^l F_{ij,-}(z,X)V_{kl}(y,X)$

Or else, such an end point might be the special X^2A vertex from which our spider began its journey. The term corresponding to this later possibility is:

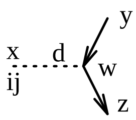
The context:	The vertex $R6$:	The formula for $R6$:
		$F_{ij,-}(z,X)\omega^k \dot{X}^l F_{kl,-}(X,y)$

8.3.4 The spider meets a ghost

As usual, we first have a contact contribution from right before the spider-ghost meeting:

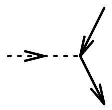
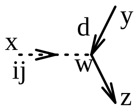
The context:	The vertex $R7$:	The formula for $R7$:
		$\frac{-1}{4\pi}\int_{\mathbf{R}^3} du\,\epsilon^{lmn}\epsilon_{mnp}(\partial_u^p G(u,w))G(u,z)$ $\cdot F_{ij,-}(x,u)V_{kl}(y,u)$ $= \frac{-1}{2\pi}\int_{\mathbf{R}^3} du\,G(u,z)F_{ij,-}(x,u)$ $\cdot V_{kl}(y,u)\partial_u^l G(u,w)$

Then we also get a 'dead end' contribution

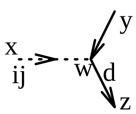


$$\longrightarrow \frac{1}{2\pi}\int_{\mathbf{R}^3} dw\,G(y,w)\left(\frac{\partial}{\partial w^k}F_{ij,-}(x,w)\right)\partial_w^k G(w,z), \tag{8.16}$$

which can be expanded further by integrating w by parts and using Leibnitz' rules similarly to what was done in (8.11). There are two resulting terms. The first one is when Leibnitz' rule instructs us to turn left in (8.16). In this case there isn't really much that we can do, so we just leave the resulting term as it is:

The context:	The vertex $R8$:	The formula for $R8$:
		$\frac{-1}{2\pi}\int_{\mathbf{R}^3} dw\,F_{ij,-}(x,u)\,(\partial_k^w G(y,w))$ $\cdot\partial_w^k G(w,z)$

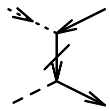
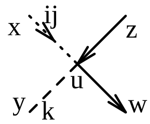
The second possibility is that Leibnitz' rule instructs us to turn right in (8.16). In this case we get



$$\longrightarrow \frac{-1}{2\pi}\int_{\mathbf{R}^3} dw\,F_{ij,-}(x,u)G(w,y)\partial_k^w\partial_w^k G(w,z)$$

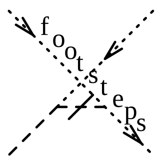
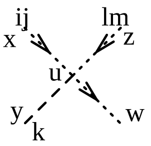
$$= \int_{\mathbf{R}^3} dw\,F_{ij,-}(x,u)G(w,y)\delta(w-z).$$

Integrating w and bringing into sight the $\bar{c}Ac$ vertex at the z side of the w-z propagator, we get the following contribution to $\delta\int\mathcal{E}(D)$:

The context:	The vertex $R9$:	The formula for $R9$:
		$\frac{1}{2\pi}\int_{\mathbf{R}^3} dw\, G(u,z)F_{ij,-}(x,u)V_{kl}(y,u)$ $\cdot\partial_u^l G(u,w)$

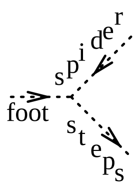
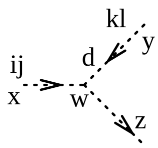
8.3.5 The spider meets his own footsteps

The contact contribution from right before the meeting is:

The context:	The vertex $R10$:	The formula for $R10$:
		$\frac{-i}{8\pi}\int_{\mathbf{R}^3} du\,\epsilon^{nst}\epsilon_{str}\epsilon^{rpq}F_{pq,-}(u,w)$ $\cdot F_{lm,-}(y,u)F_{ij,-}(x,w)V_{kn}(y,u)$ $=\frac{-i}{4\pi}\int_{\mathbf{R}^3} du\,\epsilon^{npq}F_{pq,-}(u,w)$ $\cdot F_{lm,-}(y,u)F_{ij,-}(x,w)V_{kn}(y,u)$

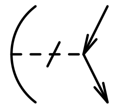
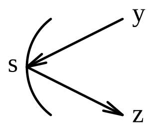
In the above diagram, the 'footsteps' are assumed to be the dotted $(\cdots\cdots>\cdots\cdots)$ propagators connecting x to u and u to w, and the spider comes back to the area from the direction of z. This explains the 'twist' in the context column.

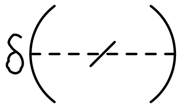
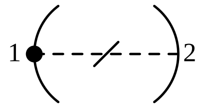
There is also the 'dead end' contribution, which we simply leave as it is:

The context:	The vertex $R11$:	The formula for $R11$:
		$\frac{-i}{4\pi}\int_{\mathbf{R}^3} dw\, F_{ij,-}(x,w)\epsilon^{mnp}F_{np,-}(w,z)$ $\cdot\partial_m^w F_{kl,-}(y,w)$

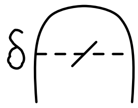
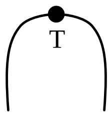
8.3.6 The journey ends before it really started

The spider's journey might end before it really gets going if he has a too short chain of gauge $(-\ -\ -\ -)$ propagators to travel on — namely, if that chain is of length 1 — namely, if the spider starts on an X^2A vertex that is connected via a gauge $(-\ -\ -\ -)$ propagator to anything *but* an A^3 vertex. The three possibilities are:

The context:	The vertex $R12$:	The formula for $R12$:
		$-iG(y,X)\omega^i\dot{X}^j\epsilon_{ijk}\ \partial_w^k G(w,z)\Big\vert_{w=X}$

The context:	The vertex S:	The formula for S:
		$\epsilon_{ijk}\omega^i\dot{X}_1^j\dot{X}_2^k\delta(X_1-X_2)$

and

The context:	The vertex T:	The formula for T:
		$\epsilon_{ijk}\omega^i\dot{X}^j\dot{X}^k\delta(X-X)$

Notice that the last two contributions differ only by the separation between the two ends of the gauge propagator being treated. In T these two ends are assumed to be adjacent, while in S they are assumed to be separated by some other X^2A vertices.

8.4 cancellations

In the previous section we computed $\delta\mathcal{W}_m$ and found that it is given by a sum of 14 types of contributions: $R1$-$R12$, S, and T. In this section we will see that these contributions all cancel each other, and therefore $\delta\mathcal{W}_m = 0$. Let $\mathcal{R}n$ denote the total contribution to $\delta\mathcal{W}_m$ that comes from diagrams of type Rn, $\mathcal{S}$ denote the total contribution of type S, and $\mathcal{T}$ denote the total contribution of type T.

Proposition 8.4.1

$$\mathcal{R}1 + \mathcal{R}2 = 0. \tag{8.17}$$

Proof The identity

$$\epsilon_{klm}\epsilon^{mnp} = \delta_k^n\delta_l^p - \delta_k^p\delta_l^n \tag{8.18}$$

shows that vertices of type $R2$ are, in fact, precisely the negatives of to vertices of type $R1$, while the context columns in the definitions of these two vertices shows that $R1$ comes with weight $C(T)-C(U)$, and that $R2$ comes with weight $C(S)$. The STU identity (8.3) concludes the proof.

□

Proposition 8.4.2

$$\mathcal{R}3 = 0$$

Proof Diagrams of type $R3$ come with weights $C(I)$, $-C(H)$, or $C(X)$, as can be read from the context column in the definition of $R3$. The IHX identity (8.2) shows that these weights cancel each other.

$\square$

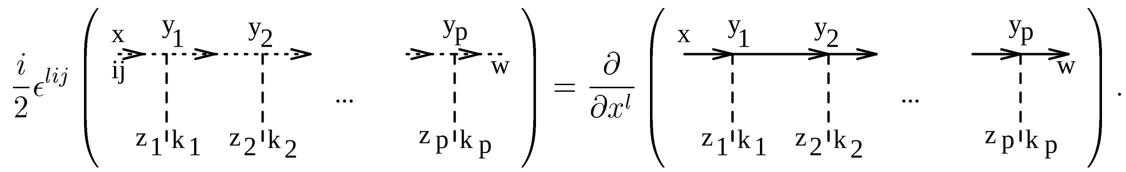
Proposition 8.4.3

$$\mathcal{R}4 + \mathcal{R}5 = 0$$

Proof Similarly to (8.17) this identity follows from the STU identity (8.3).

$\square$

The proofs of the following three propositions rely on the observation that a chain of dotted ($\cdots\cdots>\cdots\cdots$) propagators connected by F^2A vertices is essentially equivalent to a chain of ghost ($\longrightarrow$) propagators connected by $\bar{c}Ac$ vertices:



(8.19)

This identity is an immediate consequence of the definition of the ghost propagator (8.7), the definition of the dotted propagator (8.13), the definitions of the F^2A and $\bar{c}Ac$ vertices ((8.14) and (8.5)), and the identity

$$\epsilon^{lnp}\epsilon_{npq} = 2\delta^l_q.$$

Proposition 8.4.4

$$\mathcal{R}8 + \mathcal{R}11 = 0.$$

Proof Immediate from the dotted-ghost relation (8.19), the definition of the $R8$ and $R11$ vertices, and the fact that the diagrams of type $\mathcal{R}8$ have one more ghost loop than their counterparts of type $\mathcal{R}11$ and therefore they get opposite signs from (2.5).

$\square$

Proposition 8.4.5

$$\mathcal{R}6 + \mathcal{R}12 = 0.$$

Proof Immediate from (8.19), (8.18), (2.5), and the STU relation (8.3).

$\square$

Proposition 8.4.6

$$\mathcal{R}7 + \mathcal{R}9 + \mathcal{R}10 = 0.$$

Proof Immediate from the dotted-ghost relation (8.19), from (2.5), and from the IHX relation (8.2).

$\square$

Proposition 8.4.7

$$\mathcal{S} = 0.$$

Proof We just have to remember that the points 1 and 2 in the definition of the term S are always distinct, and therefore $\delta(X_1 - X_2) = 0$.

□

Remark This proof is actually more interesting when it breaks down — when the knot X is deformed in such a way that a self-intersection is created. In this case the points 1 and 2 are not necessarily distinct, $\delta(X_1 - X_2)$ can be non-zero, and when it is non-zero we get a skein-like relation similar to Vassiliev's relation (9.23). It is exactly this term S that assures that $\mathcal{W}_m(X)$ is a *non-trivial* knot invariant!

Proposition 8.4.8 *If one is willing to be a bit naive,*

$$\mathcal{T} = 0.$$

Proof The formula for the term T is

$$\det\left(\dot{X}\middle|\,\omega\,\middle|\dot{X}\right)\delta(X - X).$$

If one is willing to be a bit naive, then the determinant in the first part of this formula, $\det(\dot{X}|\omega|\dot{X})$, vanishes because it has two equal columns and this cancels the infinity of $\delta(X - X)$.

□

Remark Actually, proposition 8.4.8 is *blatantly false.* $0 \cdot \infty = 0$ doesn't make much mathematical sense as it stands, particularly when the ∞ is such a 'large' ∞ — it is a three dimensional δ-function integrated on just a line! So clearly, more care needs to be taken when considering the vertex T. This is essentially what is done in section 3.3, where it is shown that the *failure* of proposition 8.4.8 is proportional to the total torsion τ of X. I believe that the same "correction" procedure that was used there — subtraction of a certain multiple of τ — can be used in the higher loop case introducing a framing dependence to $\mathcal{W}_m$. This is yet to be proven.

Either way, whether by choosing to be naive or by believing that the failure of proposition 8.4.8 can be corrected as in section 3.4, propositions 8.4.1-8.4.8 prove that

$$\delta\mathcal{W}_m(X) = \mathcal{R}1 + \ldots + \mathcal{R}12 + \mathcal{S} + \mathcal{T} = 0,$$

and therefore $\mathcal{W}_m(X)$ should be a knot invariant.

□

Chapter 9

The Lie-algebraic weights of Feynman diagrams

9.1 Introduction

The purpose of this chapter is to introduce a certain combinatorial-algebraic problem, discuss its significance to knot theory (and to a lesser extent, to quantum field theory), and present some solutions of it. The most general solution to this problem has not yet been found, and finding it is likely to lead to the discovery of new knot and link invariants.

In this chapter, the words *closed diagram* will always refer to a graph made of a certain number of directed ellipses (called *Wilson loops*) marked by the natural numbers $1, \ldots, I$, and a certain number of dashed lines (called *propagators*). The propagators and the Wilson loops are allowed to meet in two types of vertices — one type (called *type* $R^2\mathcal{G}$) in which a propagator ends on one of the Wilson loops, and another (called *type* $\mathcal{G}^3$) connecting three propagators. We assume that the second kind of vertices are *oriented* — that one of the two possible cyclic orderings of the three propagators meeting in such a vertex is specified. The *order* of such a diagrams will be half the total number of vertices in it.

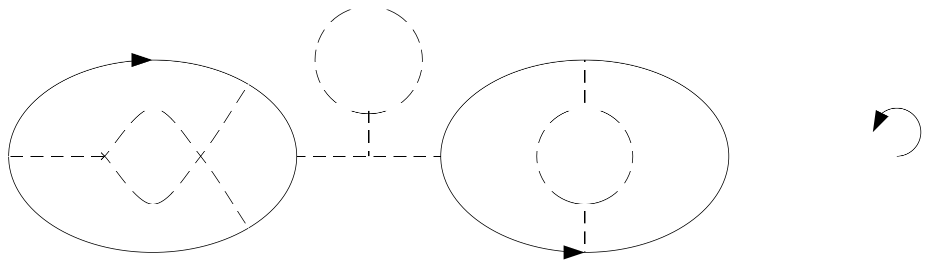


Figure 9.1. An example for a closed diagram of order 6.

Figure 9.1 is an example for such a diagram with $I = 2$. In this figure (as in the rest of this chapter) each of the vertices is oriented *counterclockwise* ($\circlearrowleft$). This convention means that the two diagram parts in figure 9.2 are *not* equivalent. Also, remember that our diagrams are not allowed to have higher than cubic vertices. It is therefore implicitly understood that when four or more lines meet at the same point, that point is not a vertex and those lines pass each other without "interaction".

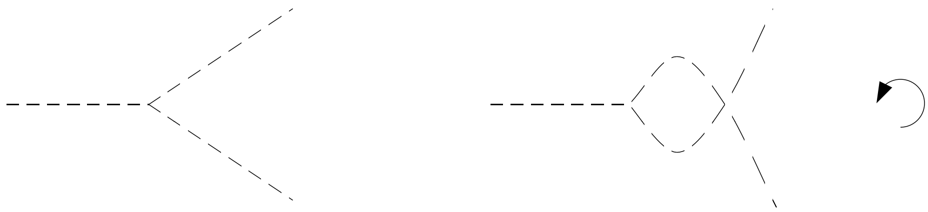


Figure 9.2. Two diagram parts which differ only by the orientation of one of their vertices.

We will be looking for assignments $D \to C(D)$ that assign a weight $C(D)$ inside some pre-chosen Abelian group to each diagram D, and satisfy the following two relations:

The "IHX" relation: Let the diagrams I, H, and X be identical outside a small domain, inside of which they look as in figure 9.3. Then their weights are expected to satisfy

$$C(I) = C(H) - C(X).$$

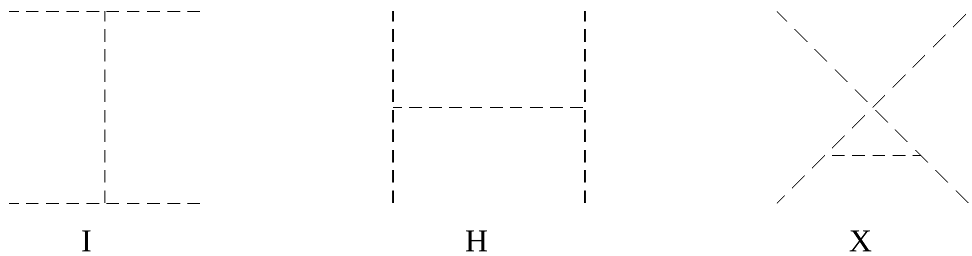


Figure 9.3. The diagrams I, H, and X.

The "STU" relation: Let the diagrams S, T, and U be identical outside a small domain, inside of which they look as in figure 9.4. Then their weights are expected to satisfy

$$C(S) = C(T) - C(U).$$

Main problem Find all such assignments C.

Such assignments will be called *weight systems*.

There are very good reasons to believe that each weight system will give rise to a link invariant. When one considers the perturbative expansion of the Chern-Simons quantum field theory as described here, one encounters diagrams much like the above. The diagrams in the Chern-Simons theory correspond to integrals, and I have shown in chapter 8 that (assuming some convergence which is yet to be proven) these integrals summed with 'correct' weights add up to give link invariants. The

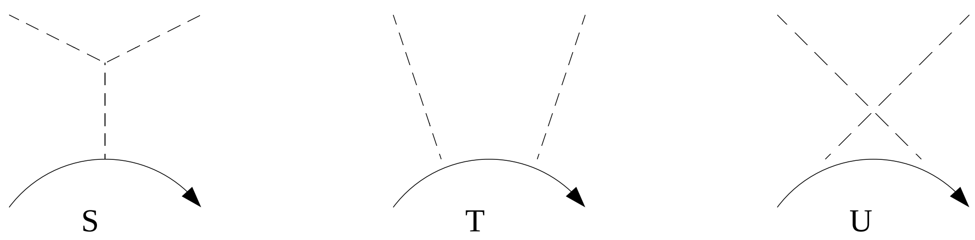


Figure 9.4. The diagrams S, T, and U.

word 'correct' in the previous sentence means exactly "satisfying the relations IHX and STU". In chapter 4 I have carried out this program for the diagrams of order ≤ 2, and in [42, 43] Witten has shown that the HOMFLY polynomial [23] can be derived from the Chern-Simons quantum field theory, and therefore can probably be re-derived using our techniques. The weight system C that should correspond to the HOMFLY polynomial is presented in section 9.5. I don't know which are the knot invariants corresponding to most of the other weight systems presented in this chapter, and I do not know whether there are further weight systems beyond those presented here.

As was (implicitly) shown in [42] and discussed in this thesis from the perturbative point of view, to each weight system should correspond a three-manifold invariant as well.

In section 9.6 a second relation, due to Vassiliev [41] and Birman-Lin [10], between those weight systems and knot theory is discussed.

9.2 The method

Let $\mathbf{F}$ be a field, and let D be a closed diagram. I will now show how, given some Lie algebraic data, we can associate an element $C_{\mathcal{G}}(D)$ of $\mathbf{F}$ to D. Of course, the construction below is precisely the 'Lie-algebraic' part of the construction in chapter 2.

Let $\mathcal{G}$ be a finite dimensional Lie algebra over the field $\mathbf{F}$, $R_1, \ldots, R_I$ a list of finite dimensional representations of $\mathcal{G}$ (one for each Wilson loop in D) of dimensions $d_1, \ldots, d_I$, and let $\mathfrak{tr}$ be a non-degenerate $\mathbf{F}$-valued ad-invariant bilinear form on $\mathcal{G}\otimes\mathcal{G}$, where ad denotes the adjoint representation of the Lie algebra $\mathcal{G}$ on its underlying vector space. Let $\{\mathcal{G}_a\}$ be a basis for $\mathcal{G}$, $\{r_i^\alpha\}$ a basis of R_i, and define the tensors t_{ab}, t^{bc}, f_{ab}^c, t_{abc}, and $R_{i\alpha\beta}^\alpha$ by the following formulae:

$$\begin{aligned}
t_{ab} &= \mathfrak{tr}(\mathcal{G}_a, \mathcal{G}_b), \\
t_{ab}t^{bc} &= \delta_a{}^c, \\
[\mathcal{G}_a, \mathcal{G}_b] &= f_{ab}^c \mathcal{G}_c, \\
t_{abc} &= f_{ab}^d t_{dc}, \\
R_i(\mathcal{G}_a) r_i^\alpha &= R_{ia\beta}^\alpha r_i^\beta.
\end{aligned}$$

To define $C_{\mathcal{G}}(D)$, first mark every Wilson loop segment in D by a greek letter

$\alpha, \beta, \ldots$, and every end of every propagator by a small letter in the English alphabet — a, b, ...

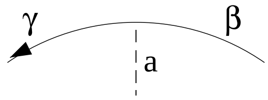
Figure 9.5. An unmarked diagram and a marked diagram.

I will now describe how to construct a certain algebraic expression out of D and its marking:

1. To each type $\mathcal{G}^3$ vertex in D associate a $t_{\ldots}$ symbol with the $\cdots$ replaced by the letters marking that vertex, picking those letter in an order consistent with the orientation of the vertex. Using the invariance of t_{ab} it is easy to check that $t_{abc} = t_{bca} = t_{cab}$, and so the particular order chosen is immaterial.

2. To each propagator in D associate a $t^{\cdot\cdot}$ symbol with the dots replaced by the letters marked at the ends of that propagator.

3. To each type $R^2\mathcal{G}$ vertex associate an $R^{\cdot}_{\cdot\cdot}$ symbol with the dots replaced by the letters marking that vertex, as in the figure below:



$$\longrightarrow R^{\beta}_{a\gamma}$$

4. Take the product of all the above mentioned $t_{\ldots}$, $t^{\cdot\cdot}$, and $R^{\cdot}_{\cdot\cdot}$ symbols.

5. Sum over $\alpha, \beta, \ldots$, and a, b, ..., and call the result $C_{\mathcal{G}}(D)$.

For example, if D is the diagram in figure 9.5, then (summation understood)

$$C_{\mathcal{G}}(D) = t_{a'b'c'} t^{a'a} t^{b'b} t^{c'c} R^{\beta}_{a\gamma} R^{\gamma}_{b\alpha} R^{\alpha}_{c\beta} \tag{9.1}$$

Well-definedness We will now check that $C_{\mathcal{G}}(D)$ is independent of the choices of bases that were made. Clearly, $C_{\mathcal{G}}(D)$ is independent of the choice of $\{r^\alpha\}$ — as is demonstrated in (9.1) the representation R appears only through matrix traces of the form

$$tr\ R(\mathcal{G}_a)R(\mathcal{G}_b)R(\mathcal{G}_c).$$

Suppose that $\{\bar{\mathcal{G}}_{\bar{a}}\}$ is a different basis of $\mathcal{G}$. One can define $\bar{t}^{\bar{a}\bar{b}}$, $\bar{t}_{\bar{a}\bar{b}\bar{c}}$, and $\bar{R}^{\alpha}_{\bar{a}\beta}$ with respect to this new basis, and use these tensors to define $\bar{C}_{\mathcal{G}}(D)$. We will show now that $\bar{C}_{\mathcal{G}}(D) = C_{\mathcal{G}}(D)$. The two bases are related by some linear transformation — that is to say, there exists a matrix $\{M_a^{\bar{a}}\}$ for which

$$\bar{\mathcal{G}}_{\bar{a}} = M_{\bar{a}}^{a} \mathcal{G}_a$$

One can check rather easily that the new tensors are given by the old ones through the following formulae:

$$\begin{aligned}
\bar{t}_{\bar{a}\bar{b}} &= M_{\bar{a}}^{a} M_{\bar{b}}^{b} t_{ab} \\
\bar{t}^{\bar{a}\bar{b}} &= (M^{-1})_a^{\bar{a}} (M^{-1})_b^{\bar{b}} t^{ab} \\
\bar{t}_{\bar{a}\bar{b}\bar{c}} &= M_{\bar{a}}^{a} M_{\bar{b}}^{b} M_{\bar{c}}^{c} t_{abc} \\
\bar{R}_{\bar{a}\beta}^{\alpha} &= M_{\bar{a}}^{a} R_{a\beta}^{\alpha}
\end{aligned}$$

where $(M^{-1})_a^{\bar{a}}$ is the inverse matrix of $M_{\bar{a}}^{a}$. It is now easy to see that when these expressions for $\bar{t}^{\bar{a}\bar{b}}$, $\bar{t}_{\bar{a}\bar{b}\bar{c}}$, and $\bar{R}_{\bar{a}\beta}^{\alpha}$ are combined together to form $\bar{C}_{\mathcal{G}}(D)$, every matrix $(M^{-1})_a^{\bar{a}}$ cancels every $M_{\bar{a}}^{a}$.

9.3 Relations between the $C_{\mathcal{G}}(D)$'s

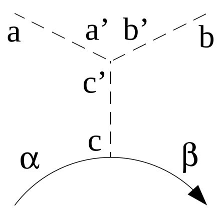
9.3.1 Tensors and relations between them

So far, we used the fact that the tensors that went into the construction of $C_{\mathcal{G}}(D)$ came from a Lie algebra and satisfied certain relations only in a very mild way — in checking that $t_{abc} = t_{bca} = t_{cab}$. We will now see what relations among the $C_{\mathcal{G}}(D)$'s can be deduced from the relations that t^{ab}, t_{abc}, and $R_{a\beta}^{\alpha}$ are known to satisfy.

First, a slight generalization. Using more or less the same procedure as before we can assign to every non-closed diagram D, which is allowed to have propagators with "free" ends and non-closed Wilson lines, a tensor

$$\mathcal{T} = \mathcal{T}(D) \in \mathcal{G}^{\otimes n} \otimes \bigotimes_{i=1}^{J} \left(R_i \otimes \bar{R}_i \right). \tag{9.2}$$

Here n is the number of propagators with *free* ends, $R_1, \dots R_J$ are the representations corresponding to the non-closed Wilson lines, and the $\bar{R}_i$'s are their duals. It is clear how to define $\mathcal{T}$ — one just needs to follow the same steps as in the definition of $C_{\mathcal{G}}$, and as D is not closed some of the indices will appear only once in the resulting expression and instead of being summed over these indices will serve as the indices of the tensor $\mathcal{T}$. For example:



$$\longrightarrow \qquad \mathcal{T}_{\beta}^{\alpha,ab} = \sum_{a',b',c',c} t^{aa'} t^{bb'} t^{cc'} t_{b'a'c'} R_{c\beta}^{\alpha} \in \mathcal{G}^{\otimes 2} \otimes R \otimes \bar{R}$$

Claim 1 *The two diagrams in figure 9.2 correspond to tensors which are the negatives of each other.*

Proof The is simply the fact that the Lie bracket is anti-symmetric.

$\square$

Claim 2 *Let the diagrams S, T, and U be as in figure 9.4. Then the tensors corresponding to them satisfy:*

$$\mathcal{T}(S) = \mathcal{T}(T) - \mathcal{T}(U) \tag{9.3}$$

Proof This is simply the fact that R is a representation. That is, that $R([\mathcal{G}_a, \mathcal{G}_b]) = R(\mathcal{G}_a)R(\mathcal{G}_b) - R(\mathcal{G}_b)R(\mathcal{G}_a)$.

$\square$

Claim 3 *Let the diagrams I, H, and X be as in figure 9.3. Then the tensors corresponding to them satisfy:*

$$\mathcal{T}(I) = \mathcal{T}(H) - \mathcal{T}(X) \tag{9.4}$$

Proof Translating I, H, and X into their corresponding tensors, it is easy to see that this is simply the Jacobi identity! (In fact, this claim can be regarded as a particular case of the previous one, asserting that the adjoint action of a Lie-algebra on itself is a representation).

$\square$

9.3.2 Sewing

Given two open diagrams A and B and a (partial) correspondence φ between their open ended lines which sends a propagator to a propagator and an ingoing (outgoing) Wilson line to an outgoing (ingoing) Wilson line labeled by the same representation, one can define their *join* $A\#B$ to be the diagram obtained by sewing the external lines of A with those of B according to the correspondence φ. It is also possible to *sew* $\mathcal{T}(A)$ to $\mathcal{T}(B)$ by contracting their indices as dictated by φ, (using t_{ab} to lower the propagator indices). It is clear that the resulting $\mathcal{T}(A)\#\mathcal{T}(B)$ will equal $\mathcal{T}(A\#B)$. In particular, if $A\#B$ is a closed diagram, then $C_{\mathcal{G}}(A\#B) = \mathcal{T}(A)\#\mathcal{T}(B)$. (See figure 9.6).

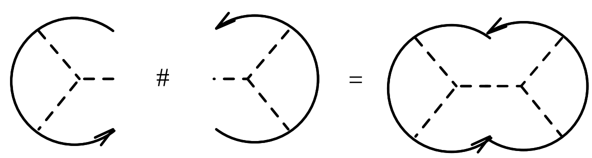


Figure 9.6. Sewing two diagrams.

Thus (9.4) and (9.3) can be used to derive relations between closed diagrams — (9.4) says that if three diagrams $\bar{I}$, $\bar{H}$ and $\bar{X}$ are identical outside of a small domain in which they look like the diagrams I, H, and X of figure 9.3, then they satisfy

$$C_{\mathcal{G}}(\bar{I}) = C_{\mathcal{G}}(\bar{H}) - C_{\mathcal{G}}(\bar{X}). \tag{9.5}$$

Similarly, (9.3) implies

$$C_{\mathcal{G}}(\bar{S}) = C_{\mathcal{G}}(\bar{T}) - C_{\mathcal{G}}(\bar{U}). \tag{9.6}$$

The last two relations show that $D \to C_{\mathcal{G}}(D)$ is a weight system in the sense of section 9.1.

Lemma 9.3.1 *For any open diagram D, $\mathcal{T} = \mathcal{T}(D)$ is an invariant tensor (with respect to the natural action of $\mathcal{G}$ on each of the components in (9.2)).*

Proof The reason why this lemma is true, is that $\mathcal{T}$ can be seen as the contraction of a collection of invariant tensors — the $t_{...}$, the $t^{..}$ and the $R^{.}_{..}$ are all invariant. This statement can be translated into a combinatorial invariance proof. I will just sketch this proof here, and supplement this sketch with a simple example — figure 9.7.

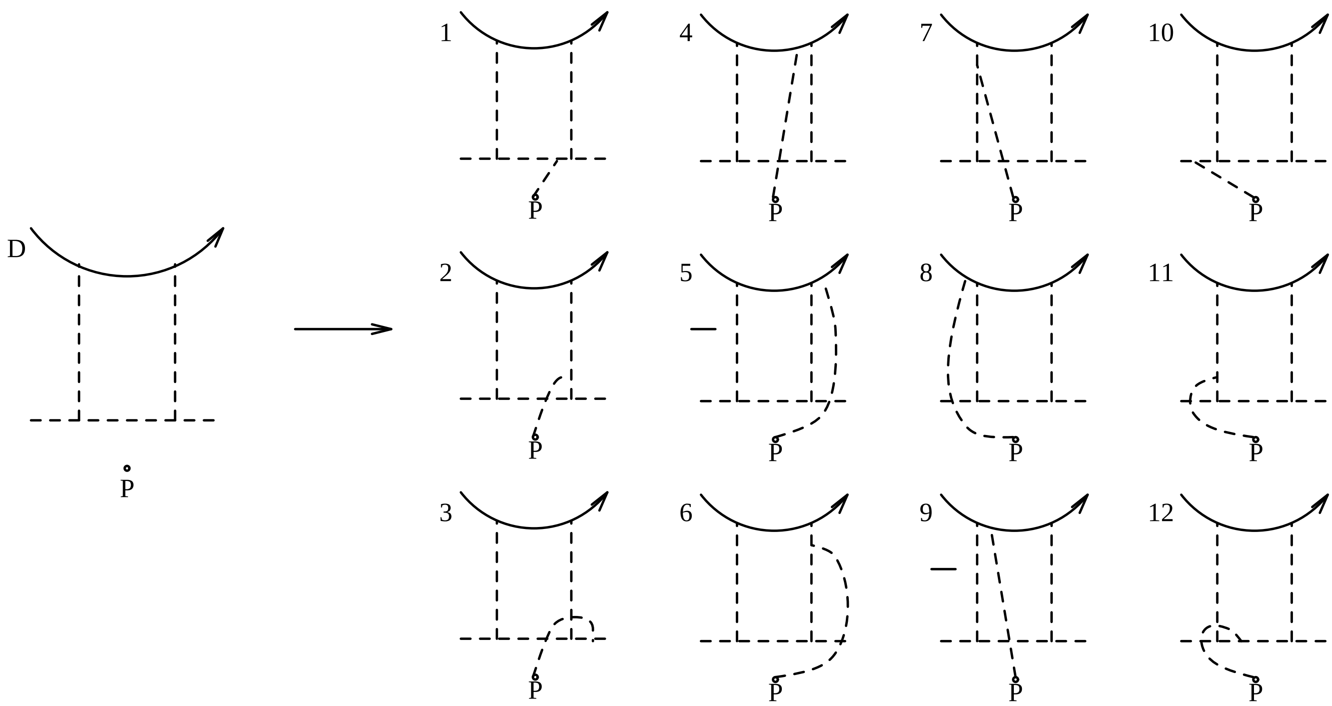


Figure 9.7. A simple invariance proof — the tensor $\mathcal{D}$ is the sum of **1-12**. Relation IHX shows that $\mathbf{1}+\mathbf{2}+\mathbf{3} = \mathbf{10}+\mathbf{11}+\mathbf{12} = 0$, relation STU shows that $\mathbf{4}+\mathbf{5}+\mathbf{6} = \mathbf{7}+\mathbf{8}+\mathbf{9} = 0$, claim 1 shows that $\mathbf{1}+\mathbf{12} = \mathbf{2}+\mathbf{6} = \mathbf{7}+\mathbf{11} = 0$, and $\mathbf{4}+\mathbf{9} = 0$ by the choice of signs. It follows that $\mathbf{3}+\mathbf{5}+\mathbf{8}+\mathbf{10} = 0$. This is exactly the fact that $\mathcal{T}$ is an invariant tensor.

For simplicity, I will disregard $\pm$ signs here. Say D has n internal vertices. Pick a point P outside of D and consider the $3n$ diagrams obtained by connecting P using a propagator to each of the three lines emanating from each of the n vertices in D. Let $\mathcal{D}$ be the sum of the tensors corresponding to these $3n$ diagrams. Each internal line in D has two terms corresponding to it in $\mathcal{D}$ coming from the two vertices at the ends of that line, and with the proper choice of signs these two terms exactly cancel. The only diagrams that still contribute to $\mathcal{D}$ are those in which P is connected to an external line, and, if P is marked by a, *these are exactly the diagrams that represent the variation of D with respect to $\mathcal{G}_a$.*

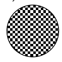
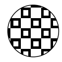
On the other hand, the relations (9.4) and (9.3) show that each group of three diagrams made by connecting P to the three lines emanating from a single propagator corresponds to tensors that add up to 0. $\mathcal{D}$ is just a sum of such groups, and this concludes the proof. (See figure 9.7).

□

Remark The behavior of $D \to \mathcal{T}(D)$ under sewing means that we've actually defined a topological Quantum Field Theory of dimension 1, satisfying Segal's axioms (see [4, 46]). Lemma 9.3.1 shows that the vector space assigned by our QFT to $n + 2J$ points, n of which labeled '$\mathcal{G}$', J labeled $R_1, \dots R_J$, and J labeled $\bar{R}_1, \dots \bar{R}_J$, is the space of invariant tensors in

$$\mathcal{G}^{\otimes n} \otimes \bigotimes_{i=1}^{J} \left(R_i \otimes \bar{R}_i\right).$$

Every diagram D with $n+2J$ free ends (of the appropriate kinds) gives a vector $\mathcal{T}(D)$ in that vector space.

Lemma 9.3.2 *If the representation R is irreducible, the factorization property illustrated in figure 9.8 holds. (In that figure, the blobs*  *and*  *simply represent arbitrary subdiagrams with an arbitrary number of connections to the Wilson loop).*

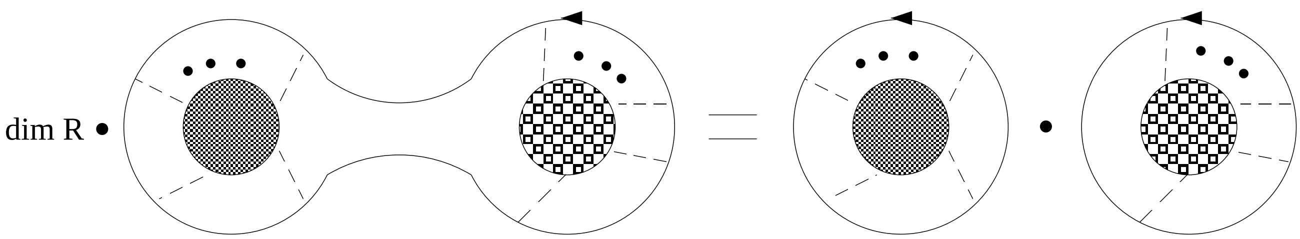


Figure 9.8. The factorization property.

Proof Clearly, the two sides of the equation in figure 9.8 represent two ways of contracting the tensors $A^{\alpha}{}_{\beta}$ and $B^{\beta}{}_{\alpha}$ corresponding to the two open diagrams obtained by removing the "bridge" in the left hand side of that equation. But from lemma

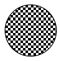
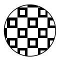
9.3.1 and the irreducibility of R it follows that A and B must be multiples of the identity matrix:

$$A^{\alpha}{}_{\beta} = a\delta^{\alpha}{}_{\beta} \quad ; \quad B^{\beta}{}_{\alpha} = b\delta^{\beta}{}_{\alpha}.$$

This reduces figure 9.8 to the trivial assertion

$$da\delta^{\alpha}{}_{\beta}b\delta^{\beta}{}_{\alpha} = a\delta^{\alpha}{}_{\alpha}b\delta^{\beta}{}_{\beta}.$$

□

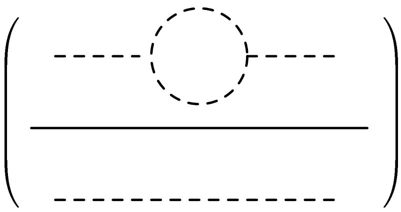
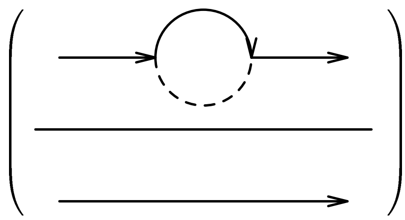
Remark taking the blobs  and  to be empty shows that it's natural to define $C_{\mathcal{G}}($  $) = \dim R = d$.

9.4 Evaluation of some diagrams for simple algebras

In this section $\mathcal{G}$ will be a simple Lie algebra over the real or complex field, and R will be an irreducible representation of $\mathcal{G}$. In this context, it is possible to evaluate some diagrams in a relatively simple way.

The key point is that under the above conditions, the spaces of invariant tensors in $\mathcal{G} \otimes \mathcal{G}$ and in $R \otimes \bar{R}$ are both one-dimensional, and therefore one can speak of 'ratios' of invariant tensors in $\mathcal{G} \otimes \mathcal{G}$ or in $R \otimes \bar{R}$.

Definition 1 *The constants r and g are given by the following ratios*[1]*:*

$g =$  $r =$  . (9.7)

(Notice that by lemma 9.3.1 the above tensors are all invariant).

In the following few lines, we see how the relations from the previous section can be used to evaluate $C_{\mathcal{G}}$ for all closed diagrams with a single Wilson loop and orders smaller than three. For brevity, we omit the symbol $C_{\mathcal{G}}$ below.

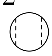
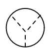
 $= d$ — by the remark after lemma 9.3.2 (9.8)

 $= r$  $= dr$ — by (9.7) and (9.8) (9.9)

 $= g$  $= dgr$ — by (9.7) and (9.9) (9.10)

 $= r$  $= dr^2$ (9.11)

 $= \frac{1}{2}\Big($  $-$  $\Big) = \frac{1}{2}$  $= \frac{1}{2}dgr$ — by (9.3) (9.12)

 $=$  $-$  $= dr\left(r - \frac{1}{2}g\right)$ (9.13)

[1]Using the notation of chapter 5, $g = c_2(\mathcal{G})$ and $r = c_2(R)$.

Similarly:

 $= dgr^2$

 $= dr^3$

 $= dr^3$

 $= dr^2(r - \frac{1}{2}g)$

 $= \frac{1}{2}dgr^2$

 $= \frac{1}{2}dg^2r$

 $= \frac{1}{4}dg^2r$

 $= \frac{1}{2}dgr(r - \frac{1}{2}g)$

 $= dr(r - \frac{1}{2}g)^2$

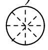
 $= dgr(r - \frac{1}{2}g)$

 $= \frac{1}{4}dg^2r$

 $= 0$

 $= dr(r - \frac{1}{2}g)(r - g)$

 $= dg^2r$

Unfortunately, there are some order four (and higher) diagrams that cannot be evaluated using these techniques. One such diagram is  .

The following table contains the values of d, g, and r for some classical Lie algebras with their defining representations (and t_{ab} taken to be the matrix trace in those representations):

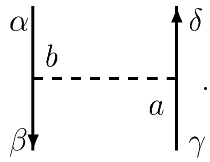
$\mathcal{G}$	R	d	g	r
$sl(N, \mathbf{C})$	$\mathbf{C}^N$	N	$2N$	$\frac{N^2-1}{N}$
$so(N, \mathbf{C})$	$\mathbf{C}^N$	N	$N-2$	$\frac{N-1}{2}$
$sp(N, \mathbf{C})$	$\mathbf{C}^{2N}$	$2N$	$2(N+1)$	$N + \frac{1}{2}$

Remark One can check that if $\mathcal{G}$ is a real Lie algebra and $\mathcal{G}_{\mathbf{C}}$ is its complexification then $C_{\mathcal{G}} \equiv C_{\mathcal{G}_{\mathbf{C}}}$. Therefore the above table can be used to evaluate d, g, and r for any of the real forms of $sl(N, \mathbf{C})$, $so(N, \mathbf{C})$, or $sp(N, \mathbf{C})$ in their defining representations.

9.5 Complete evaluation for the classical algebras

By the remark at the end of the previous section, to calculate $C_{\mathcal{G}}$ for the classical algebras (in their defining representations) it is enough to consider the four complex classical algebras.

The first step is to use relation STU repeatedly, with each usage reducing the number of $\mathcal{G}^3$ vertices by one, until we are left with a diagram D that has no $\mathcal{G}^3$ vertices. The basic building block of such diagrams is the tensor

$\mathcal{T}^{\alpha\gamma}_{\beta\delta} =$  .

This tensor will be evaluated explicitly for each of the complex classical algebras, and the results will turn out to have representations in terms of diagrams that have no propagators in them. Using this repeatedly, we are left with disjoint unions of circles which again are easy to evaluate explicitly.

I will show in detail the computations for $so(N, \mathbf{C})$, and just state the results for $gl(N, \mathbf{C})$, $sl(N, \mathbf{C})$, and $sp(N, \mathbf{C})$.

9.5.1 The algebra $so(N, \mathbf{C})$.

A convenient choice of generators for $so(N, \mathbf{C})$ are the $N \times N$ matrices M_{ij} $(i < j)$, given by

$$(M_{ij})_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta} - \delta_{i\beta}\delta_{j\alpha}.$$

That is, the ij entry of M_{ij} is $+1$, the ji entry of M_{ij} is -1, and all other entries of M_{ij} are zero. The invariant bilinear form that we pick on $so(N, \mathbf{C})$ is the matrix trace in the defining representation, and so

$$t_{(ij)(kl)} \stackrel{\text{def}}{=} tr(M_{ij}M_{kl}) = -2\delta_{ik}\delta_{jl}.$$

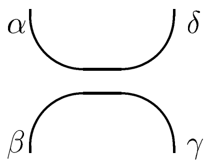
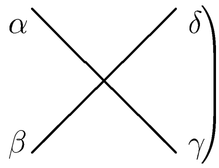
Inverting the $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$ matrix $t_{(ij)(kl)}$ we get

$$t^{(ij)(kl)} = -\frac{1}{2}\delta^{ik}\delta^{jl}, \tag{9.14}$$

and so

$$\mathcal{T}^{\alpha\gamma}_{\beta\delta} = \sum_{i<j;k<l} t^{(ij)(kl)}(M_{ij})_{\alpha\beta}(M_{ij})_{\gamma\delta}. \tag{9.15}$$

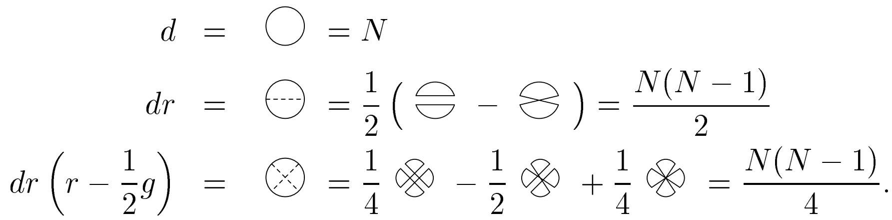
Using (9.14) and some algebraic manipulations we can simplify (9.15), and then represent it by a diagram:

$$(9.15) = \frac{1}{2}(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta}) = \frac{1}{2}\left(\right.$$  $-$  $\left.\right).\quad (9.16)$

The last thing to note is that

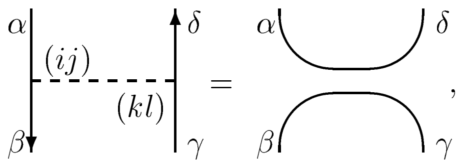
$$C_{so(N,\mathbf{C})}(k \text{ disjoint circles}) = N^k.$$

Example For $so(N, \mathbf{C})$ in its defining representation we can calculate d, r, and g using: (suppressing the '$C_{so(N,\mathbf{C})}$' symbols)



9.5.2 The algebra $gl(N, \mathbf{C})$.

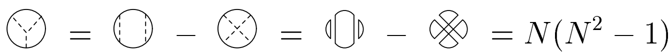
Similar considerations lead to the even simpler rule



while retaining

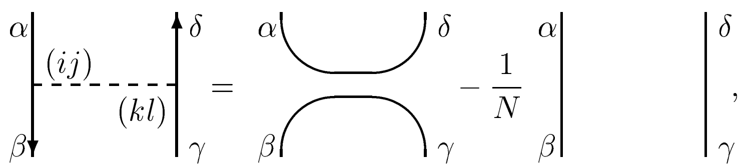
$$C_{gl(N,\mathbf{C})}(k \text{ disjoint circles}) = N^k.$$

Example For $gl(N, \mathbf{C})$ in its defining representation



9.5.3 The algebra $sl(N, \mathbf{C})$.

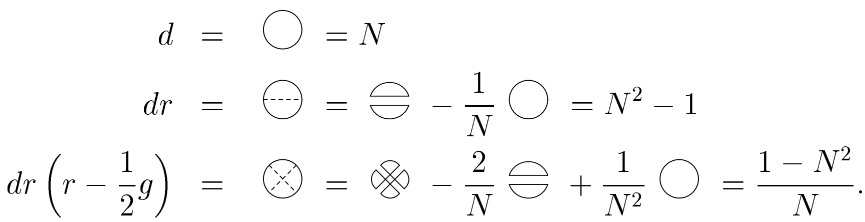
The rule here is



with the usual

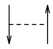
$$C_{sl(N,\mathbf{C})}(k \text{ disjoint circles}) = N^k.$$

Example For $sl(N, \mathbf{C})$ in its defining representation we can calculate d, r, and g using:

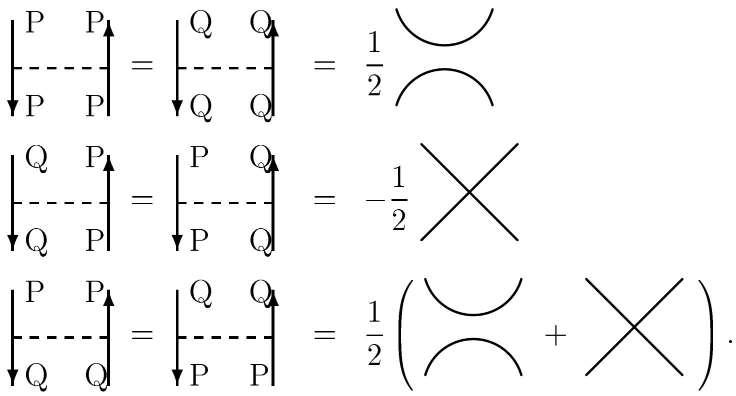


9.5.4 The algebra $sp(N, \mathbf{C})$.

This is the most complicated case. Let D be a diagram with no $\mathcal{G}^3$ vertices. The computation of $C_{sp(N,\mathbf{C})}(D)$ now proceeds in two steps:

1. Mark each Wilson loop segment in D with either the symbol P or the symbol Q, in such a way that the number of P's entering each subdiagram of D of the form  is equal to the number of P's leaving it. (Remember that the Wilson loops are *directed*).

2. Simplify D using the following rules:

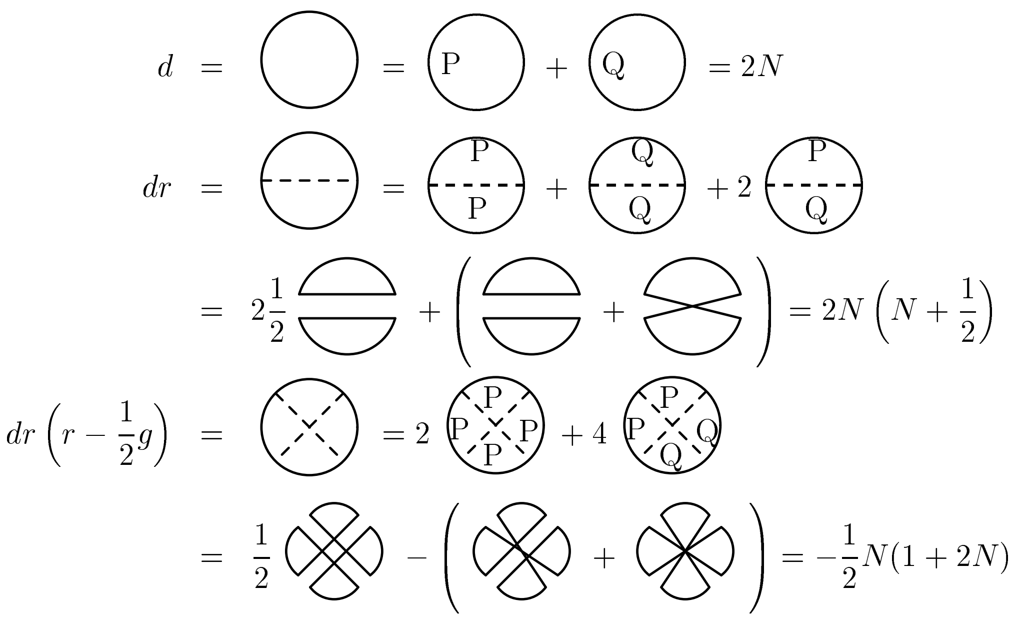


3. Similarly to the usual,

$$C_{sp(N,\mathbf{C})}(k \text{ disjoint } \textit{marked} \text{ circles}) = N^k.$$

(Notice that this time $\dim R = 2N \neq N$).

Example For $sp(N, \mathbf{C})$ in its defining representation we can calculate d, r, and g using:



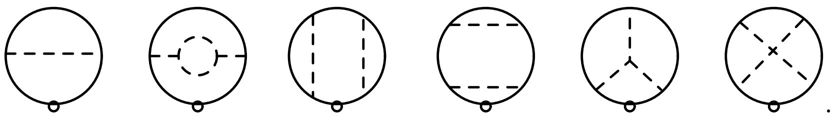
Exercise The reader might find it amusing to verify that $C_{sp(1,\mathbf{C})} \equiv C_{sl(2,\mathbf{C})}$, as expected from the isomorphism $sp(1, \mathbf{C}) \cong sl(2, \mathbf{C})$. Notice that $C_{so(3,\mathbf{C})}$ is not equal to $C_{sp(1,\mathbf{C})}$ (or $C_{sl(2,\mathbf{C})}$) because their defining representations are not the same.

9.6 Appendix: The Vassiliev knot invariants

9.6.1 Taking the logarithm

In this appendix we will assume that $\mathbf{F}$ is a field of characteristic zero and that R is an *irreducible* representation of $\mathcal{G}$.

Definition 2 *Let $\mathcal{A}$ be the vector space of (infinite) formal linear combinations (with coefficients in $\mathbf{F}$) of (graph-) isomorphism types of closed diagrams having $I = 1$, (i.e. containing exactly one Wilson loop), with a pre-chosen base point on that loop. For convenience, we will exclude the trivial diagram*  *from $\mathcal{A}$. For example, here are the six simplest generators of $\mathcal{A}$:*



In fact, $\mathcal{A}$ can be made into an algebra; the product of $A \in \mathcal{A}$ and $B \in \mathcal{A}$ will essentially be the sum of all the possible ways of merging them into a single diagram:

Definition 3 *Let A be a generator of $\mathcal{A}$, and let $a_1, a_2, \ldots, a_n$ be the list of $R^2\mathcal{G}$ vertices in A, in the order they are encountered when one travels along the loop consistently with its orientation and beginning from the base point. Let B be another generator of $\mathcal{A}$, and define $b_1, b_2, \ldots, b_k$ in the same way. Let $\mathcal{P}$ be the set of all possible linear orderings of n "a" symbols and m "b" symbols. For every $P \in \mathcal{P}$ define $[AB]_P$ to be the diagram obtained by marking a based Wilson loop with a's and b's following their order in P, and connecting diagrams A and B (minus their respective loops) to that Wilson loop following the marks in the obvious way. Finally, define*

$$A \cdot B = \sum_{P \in \mathcal{P}} [AB]_P \, .$$

For an example, see figure 9.9.

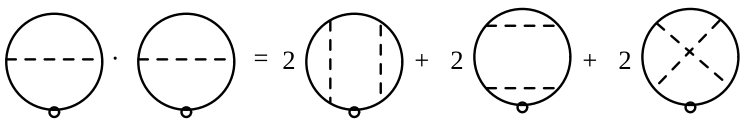


Figure 9.9. Taking the product in $\mathcal{A}$

Claim 4 *The algebra $\mathcal{A}$ is associative and commutative.*

$\square$

Now let $\mathcal{Z} \in \mathcal{A}$ be

$$\mathcal{Z} = d + \sum_{\text{generators of } \mathcal{A}} C_{\mathcal{G}}(D) \cdot D, \tag{9.17}$$

and let $\mathcal{W} \in \mathcal{A}$ be the formal logarithm of $\mathcal{Z}$,

$$\mathcal{W} = \log \mathcal{Z},$$

given by the formal power series expansion

$$\mathcal{W} \stackrel{\text{def}}{=} \log d + \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \left(\sum_D C_{\mathcal{G}}(D) \cdot D\right)^m}{m d^m}. \tag{9.18}$$

Notice that the order of $A \cdot B$ is always bigger than that of A or B, and so every diagram D appears in the above infinite sum only finitely many times, and hence $\mathcal{W}$ is well defined.

Definition 4 *Define $C'_{\mathcal{G}}(D)$ to be the coefficient of D in $\mathcal{W}$. Namely, define it by the equation*

$$\mathcal{W} = \log d + \sum_D C'_{\mathcal{G}}(D) \cdot D.$$

Remark It is easy to check that the weight of a diagram is independent of the position of its base point, which was introduced only for the sake of simplifying definition 3. Therefore, base points will be suppressed from now on.

Definition 5 *Let D be a generator of $\mathcal{A}$. A 'cyclic partition' of D will be a cyclicly ordered (that is, ordered up to a rotation) partition $\mathfrak{D} = \{D_1, D_2, \ldots, D_{k(\mathfrak{D})}\}$ of the set of all propagators of D into disjoint subsets, such that for any propagator $p \in D_i$, all the propagators connected to p by a $\mathcal{G}^3$ vertex will also be in D_i. Given such a partition, we will denote by the same letter D_i the generator of $\mathcal{A}$ obtained by reinserting the Wilson loop of D around D_i.*

Claim 5 *The weight $C'_{\mathcal{G}}(D)$ of a generator D of $\mathcal{A}$ is given in the following formula:*

$$C'_{\mathcal{G}}(D) = \sum_{\text{cyclic partitions } \mathfrak{D}} \frac{(-1)^{k(\mathfrak{D})+1}}{d^{k(\mathfrak{D})}} \prod_{i=1}^{k(\mathfrak{D})} C_{\mathcal{G}}(D_i). \tag{9.19}$$

Proof This is simply a sum over all the possible ways of writing D as a product in $\mathcal{A}$, with the coefficients taken correctly as in (9.18). The fact that we are restricting our attention to "cyclic partitions" corresponds to the factor $\frac{1}{m}$ in that equation.

$\square$

Lemma 9.6.1 *Let D be a generator of $\mathcal{A}$ which can be decomposed (in the sense of definition 5) into two parts such that:*

1. *The two parts can be separated from each other by cutting the Wilson loop of D at just two points.*
2. *At least one of the parts cannot be decomposed any further.*

In this case,

$$C'_{\mathcal{G}}(D) = 0. \tag{9.20}$$

(For an example, see figure 9.10).

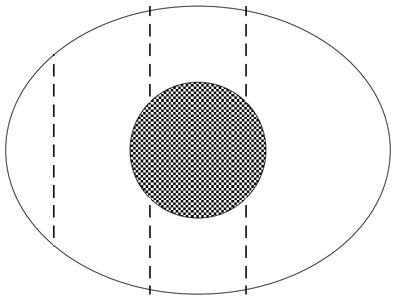


Figure 9.10. An example for a diagram with $C'_{\mathcal{G}}(D) = 0$

Proof Let $D = A \cup B$ be a diagram decomposed into two non-empty separated parts such that A cannot be be decomposed any further. Write

$$C'_{\mathcal{G}}(D) = \sum_{\text{cyclic partitions } \mathfrak{D}} c'(\mathfrak{D}) \quad ; \quad c'(\mathfrak{D}) = \frac{(-1)^{k(\mathfrak{D})+1}}{d^{k(\mathfrak{D})}} \prod_{i=1}^{k(\mathfrak{D})} C_{\mathcal{G}}(D_i).$$

We will prove (9.20) by finding a fixed point free involution $\mathfrak{D} \to \rho\mathfrak{D}$ of the set of all cyclic partitions of D for which $c'(\rho\mathfrak{D})$ is always the negative of $c'(\mathfrak{D})$.

Let $\mathfrak{D} = \{D_1, D_2, \ldots, D_{k(\mathfrak{D})}\}$ be a cyclic partition of D. There are two possibilities:

1. *A is one of the D_i's.* In this case, define $\rho\mathfrak{D}$ to be the cyclic partition obtained by adjoining A to the component of D preceding it in $\mathfrak{D}$. It is clear that $k(\rho\mathfrak{D}) = k(\mathfrak{D}) - 1$, and therefore using lemma 9.3.2 we find $c'(\rho\mathfrak{D}) = -c'(\mathfrak{D})$.
2. *A is properly contained in one of the D_i's.* We may assume that A is properly contained in D_1. Define $\rho\mathfrak{D} = \{D_1 - A, A, D_2, \ldots, D_{k(\mathfrak{D})}\}$. It is clear that $k(\rho\mathfrak{D}) = k(\mathfrak{D}) + 1$, and therefore using lemma 9.3.2 we find $c'(\rho\mathfrak{D}) = -c'(\mathfrak{D})$.

It is clear that ρ is a fixed point free involution.

□

Remark It is easy to show that the second requirement of the above lemma is superfluous — even if one of the parts of D is still decomposable one can always use relation STU to express that part as a sum of open diagrams, each of which is either 'less decomposable' or 'more separable' (i.e. can be separated in the sense of the first requirement of the above lemma into two smaller parts).

Claim 6 *The relations (9.5) and (9.6) hold for the $C'_{\mathcal{G}}(D)$'s as well:*

$$C'_{\mathcal{G}}(\bar{I}) = C'_{\mathcal{G}}(\bar{H}) - C'_{\mathcal{G}}(\bar{X}), \tag{9.21}$$

$$C'_{\mathcal{G}}(\bar{S}) = C'_{\mathcal{G}}(\bar{T}) - C'_{\mathcal{G}}(\bar{U}). \tag{9.22}$$

Proof (9.5) is a linear relation, and it is respected by each term in the sum (9.19). Therefore (9.21) holds. The same is true for (9.22), only that $\bar{T}$ and $\bar{U}$ have cyclic partitions which do not correspond to cyclic partitions of $\bar{S}$ — these are the ones in which the two propagators in T or in U of figure 9.4 appear in different components. There is natural correspondence ρ between those exceptional partitions of $\bar{T}$ and those of $\bar{U}$, and clearly $c'(\rho\mathfrak{D}) = c'(\mathfrak{D})$ for every exceptional partition $\mathfrak{D}$ of $\bar{T}$. The minus sign in (9.22) then shows that these exceptional partitions can be disregarded.

$\square$

Remark The algebra structure of $\mathcal{A}$ can be used to define an algebra structure on the space $\mathcal{C}$ of all weight systems. Let the *generating function* $\mathcal{Z}_C$ of a weight system C be as in (9.17),

$$\mathcal{Z}_C = d + \sum_{\text{generators of } \mathcal{A}} C(D) \cdot D,$$

and for $C_{1,2} \in \mathcal{C}$ define their product $C_1 \cdot C_2$ by

$$\mathcal{Z}_{C_1 \cdot C_2} = \mathcal{Z}_{C_1} \cdot \mathcal{Z}_{C_2}.$$

The above proof is essentially a verification of the fact that $\mathcal{Z}_{C_1 \cdot C_2}$ is indeed the generating function of a weight system that satisfies the relations IHX and STU.
Example The following weights can be easily computed using (9.19):

$C'_{\mathcal{G}}($  $) = r$

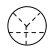
$C'_{\mathcal{G}}($  $) = gr$

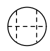
$C'_{\mathcal{G}}($  $) = \frac{1}{2}gr$

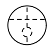
$C'_{\mathcal{G}}($  $) = -\frac{1}{2}gr$

$C'_{\mathcal{G}}($  $) = \frac{1}{2}g^2r$

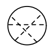
$C'_{\mathcal{G}}($  $) = \frac{1}{4}g^2r$

$C'_{\mathcal{G}}($  $) = -\frac{1}{4}g^2r$

$C'_{\mathcal{G}}($  $) = \frac{1}{4}g^2r$

$C'_{\mathcal{G}}($  $) = -\frac{1}{2}g^2r$

$C'_{\mathcal{G}}($  $) = \frac{1}{4}g^2r$

$C'_{\mathcal{G}}($  $) = \frac{1}{2}g^2r$

$C'_{\mathcal{G}}($  $) = g^2r$

It is easy to check that all the other diagrams of order ≤ 3 have a vanishing $C'_{\mathcal{G}}$.

9.6.2 The Vassiliev knot invariants

In [41] Vassiliev considered the space $\mathcal{M}$ of all the possible embeddings of the oriented circle S^1 in an oriented $\mathbf{R}^3$ as a subspace of the space of all smooth maps $S^1 \to \mathbf{R}^3$, analyzed the possible singularities of such maps, and using that information constructed a filtration of $\mathcal{M}$ and a spectral sequence that converges to its cohomology. The connected components of $\mathcal{M}$ correspond simply to oriented knot types, and therefore each element of $H^0(\mathcal{M})$ is a knot invariant. Vassiliev then uses his topological machinery to partially compute $H^0(\mathcal{M})$, and based on his machinery, Birman and Lin [10] arrived at the following properties which a numerical invariant V_i of oriented knots that comes from the i's level of Vassiliev's filtration has to satisfy:

1. V_i has an extension (which I will also denote by V_i) to an invariant of smooth immersed circles, which are allowed to have finitely many transversal self-intersection. We will call such immersed circles *embedded graphs.*

2. $V_i(\bigcirc) = 0$.

3. Overcrossings, undercrossings and self-intersections are related by:

$$V_i(\quad) - V_i(\quad) = V_i(\quad). \tag{9.23}$$

   This relation will be called the *flip* relation. (As usual in knot theory, when we write , or , we think of them as parts of bigger graphs which are identical outside of a small sphere, inside of which they look as in the figures).

4. If a graph G has *more* than i self-intersections, then $V_i(G) = 0$.

The third and fourth properties taken together imply that if a graph G has exactly i self-intersection, than $V_i(G)$ depends only on the *abstract* graph underlying G, and not on its embedding. Such a graph will be called *saturated.* A simple way of representing such a graph is by the diagram underlying it, which is obtained by drawing a circle in the plane corresponding to the parameterization of G, and connecting using a dashed line every two points of that circle which are identified in G. For an example, see figure 9.11.

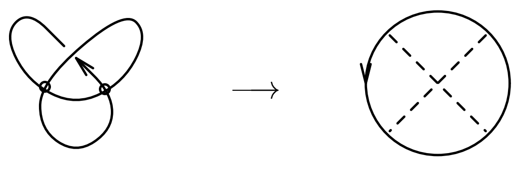
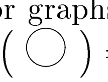


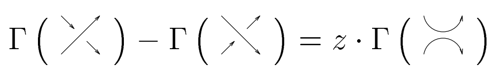
Figure 9.11. The diagram corresponding to a saturated graph with $i = 2$

Example A somewhat tautological example is easily derived from the Conway polynomial [19, 31]. Fix $i > 0$, let G an embedded graph with j self-intersections, and let

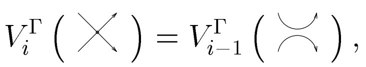
$K_1, \ldots, K_{2^j}$ to be the 2^j possible *resolutions* of G — the 2^j knots obtained by replacing each of the j self-intersections in G by either an overcrossing or an undercrossing. Let $\Gamma(K)(z)$ be the Conway polynomial of a knot K, and define

$$V_i^\Gamma(G) \stackrel{\text{def}}{=} \text{coefficient of } z^i \text{ in } \sum_{m=1}^{2^j} (-1)^{\#\text{ of undercrossings in } K_m} \cdot \Gamma(K_m)(z). \tag{9.24}$$

I have already defined V_i^Γ for graphs, and there is nothing to check for property 1. Property 2 is the fact that Γ  $= 1$ is independent of z, and property 3 is trivial from the definition (9.24). By the defining relation of the Conway polynomial



and property 3, it follows that



and this proves that if $j > i$ then $V_i^\Gamma(G) = 0$, as required in property 4. Using the results of the previous section one can check that if G is a saturated graph and D is its corresponding diagram, then $V_i^\Gamma(G)$ is equal to the coefficient of N in $C_{sl(N,\mathbf{C})}(D)$.

We saw that underlying the Vassiliev invariants there is an assignment of weights to a certain collection of diagrams, $D \to V_i(D)$, just like the assignments $C_{\mathcal{G}}$ and $C'_{\mathcal{G}}$. The Vassiliev assignments are not arbitrary — they have to satisfy certain consistency conditions: (These conditions were first written explicitly by Birman and Lin in [10])

Claim 7 *Whenever four diagrams S, E, W, and N differ only as shown in figure 9.12, their weights satisfy*

$$V_i(S) - V_i(E) = -V_i(W) + V_i(N). \tag{9.25}$$

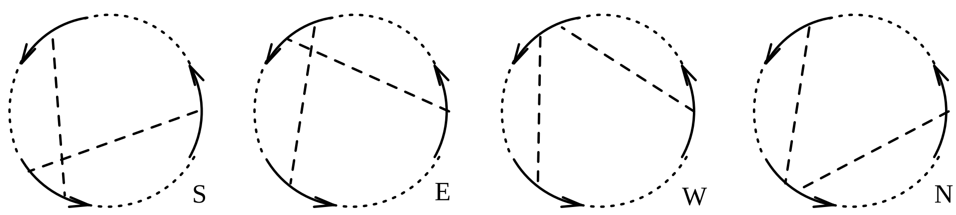


Figure 9.12. The diagrams S, E, W, and N. (The dotted arcs represent parts of the diagrams that are not shown in the figure. These parts are assumed to be the same in the four diagrams)

Proof Let SW be the almost saturated (i.e. having $i-1$ self-intersections) graph shown (partially) in figure 9.13. Pieces of the x and y axes near the origin serve as arcs in that graph, as well as a third line z' parallel to the z axis but transversing the $x-y$ plane South-West of the origin. Let NE be the same, only with the third line z' moved to transverse the $x-y$ plane North-East of the origin. There are two ways to calculate $V_i(NE)$ in terms of $V_i(SW)$ and the weights of saturated graphs using the flip relation — by moving z' from SW to NE along the two dotted paths in figure 9.13. The two ways must yield the same answer, and therefore the four saturated graphs corresponding to z' intersecting the x and y axes South, East, West and North of the origin have diagrams whose weights are related. With the sign convention of (9.23), this relation is seen to be (9.25).

□

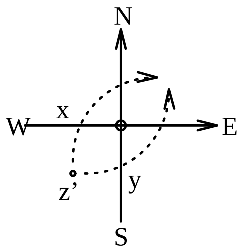
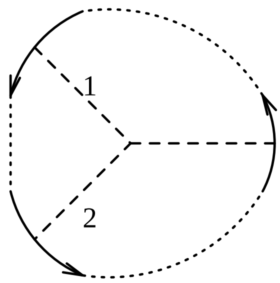
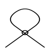


Figure 9.13. The graph SW and the two ways of getting from it to NE. Notice that z' is perpendicular to the plane and therefore appears as a single dot.

It is easy to see that the weight systems $C_{\mathcal{G}}$ and $C'_{\mathcal{G}}$ satisfy the relation (9.25). Simply use the relations (9.6) and (9.22) in two different ways (marked 1 and 2) on the diagram:



Claim 8 *(Birman-Lin) If a diagram D contains a dashed line whose endpoints on the circle are not separated from each other by an endpoint of any other line in D, then $V_i(D) = 0$.*

Proof An embedded graph G whose corresponding diagram is D would have a kink  . By the flip relation (9.23), $V_i(G) = V_i(G^o) - V_i(G^u)$, where G^o (G^u) is a version of G in which the kink was resolved to an overcrossing (undercrossing). But G^o and G^u are isotopic, and therefore $V_i(G) = 0$.

□

It is a trivial consequence of lemma 9.6.1 that The weights $C'_{\mathcal{G}}$ satisfy the relation in claim 8.

We have just solved a problem posed by Birman and Lin in [10] — to find non-trivial solutions to the relations in the last two claims.

Chapter 10

Perturbation theory beyond two loops

Following Witten [47], I will sketch here how we expect the perturbation theory of the Chern-Simons gauge theory to behave on a general three manifold and to higher order in $1/k$.

In [42, 43] Witten used very different techniques than those presented here to find a complete non-perturbative definition of the Chern-Simons gauge theory. The part of his solution that is relevant for making a comparison with the results proven here was reviewed in the previous chapter, and that comparison showed a complete agreement between the two approaches. The solution involves three subtleties that are hard to predict by just observing the definition of the theory in equation (1.2):

1. *Links have to be framed.* According to Witten's solution $\mathcal{W}(M^3, \mathcal{X}, k)$ *cannot* be defined as it is for a bare link $\mathcal{X}$, but one also has to choose a framing for each of the components of $\mathcal{X}$ and only then $\mathcal{W}(M^3, \mathcal{X}, k)$ can be defined. Its definition will then depend on the choice of the framing in a prescribed manner. This point was explained in some more detail in the chapter 5.

2. *Three-manifolds have to be framed.* According to Witten's solution $\mathcal{W}(M^3, \mathcal{X}, k)$ *cannot* be defined as it is for a bare three-manifold M^3, but one also has to choose a framing for M^3 — a choice up to homotopy of a trivialization of the tangent bundle of M^3, and only then $\mathcal{W}(M^3, \mathcal{X}, k)$ can be defined [44, 3]. (Actually, something a little less than a framing of M^3 is enough [44, 3]–it is enough, roughly speaking, to have a framing modulo torsion.) Its definition will then depend on the choice of the framing in a prescribed manner. As we were working on a flat $\mathbf{R}^3$ we have not encountered this subtlety in this paper. We can consider this subtlety and the previous one as cases of a broken symmetry — as framings do not at all appear in (1.2) it is trivialy invariant under a change of framing and this symmetry is broken in Witten's solution.

3. *Analyticity near $k = \infty$ is lost.*[1] Naively one sees that as $k \to -k$ in (1.2),

[1] Some authors [26, 27] dispute this point, which is usually referred to as "the shift in k". It is

$\mathcal{W}(M^3, \mathcal{X}, k)$ transforms to its complex conjugate. This property of $\mathcal{W}$ together with analyticity near $k = \infty$ means that we expect the even powers in the $1/k$ asymptotics of $\mathcal{W}$ to be real and the odd ones to be imaginary. This property is lost in Witten's solution as can clearly be seen from equations (6.1), (6.2), (6.3) and (6.5) in which k always appears 'shifted' by N.

All of the above mentioned subtleties seem not to appear in a naive Feynman-diagrammatic expansion of $\mathcal{W}$, and the purpose of this chapter is to show how these points probably do appear in perturbation theory after all.

Formally writing down the sums of Feynman diagrams that we expect to yield higher three-manifold and link invariants and translating them into finite dimensional integrals is routine and easy. It is also not hard to produce a *formal* invariance proof for these integrals as explained in chapter 7, ignoring the analytical difficulties arising from the divergence of those integrals. We will see below how resolving these analytical difficulties is likely to explain the three subtleties listed above.

The origin of the above mentioned analytical difficulties is the singularities Greens' functions have near the diagonal. These get milder for higher order differential operators. This suggests trying to regularize (1.2) by adding higher order terms to the Lagrangian preserving as much symmetries as possible so as not to spoil the metric independence argument of chapter 7. (Physicists call such a procedure Pauli-Villars regularization.) The main ingredient of this argument is BRST invariance (lemma 3.1), and if we wish to preserve it we can only add terms that preserve gauge invariance. The only such term of order two is the square of the norm of the curvature of the connection A and therefore we will make the replacement

$$\mathcal{L}_{tot} \to \mathcal{L}_{regularized} \stackrel{\text{def}}{=} \mathcal{L}_{tot} + \epsilon ||F_A||^2.$$

(In fact, to preserve the ellipticity of the quadratic part of $\mathcal{L}_{regularized}$ one also has to change the gauge-fixing term of $\mathcal{L}_{tot}$ and this forces changing Q slightly. Making those changes is easy and does not affect the rest of our reasoning, so we will ignore them.)

Let as now *pretend* that $\mathcal{L}_{regularized}$ gives rise to a finite perturbation theory. (Actually, this is true except for the role of a few low order subdiagrams.) What will remain of the invariance argument (7.6)?

Lemma 3.1 and lemma 3.3 will still hold because we have preserved gauge invariance, but as the additional term in $\mathcal{L}_{regularized}$ is metric dependent, lemma 3.2 will not be true any more. Instead, the variation of $\mathcal{L}_{regularized}$ under $g^{ij} \to g^{ij} + \delta g^{ij}$ will be given by

$$\delta \mathcal{L}_{regularized} = Q\Lambda + \epsilon\delta ||F_A||^2$$

and therefore in the notations of (7.6) we will have

$$\delta \langle \mathcal{O} \rangle_\epsilon = \epsilon \langle \mathcal{O} \delta ||F_A||^2 \rangle_\epsilon \tag{10.1}$$

very likely that in the context of the regularization suggested below no changes need to be made to the assertions in this paper.

where the subscript ϵ in $\langle \cdot \rangle_\epsilon$ is meant to remind us that we are taking expectation values with respect to a Lagrangian that depends on ϵ. Of course, equation (10.1) (and equations (10.2)-(10.5) as well) should be understood as an equality of perturbative asymptotic expansions, and its proof will be based on (7.6) as explained in chapter 7. If $\langle \mathcal{O} \rangle_\epsilon$ had a limit as $\epsilon \to 0$ and $\langle \mathcal{O}\delta ||F_A||^2 \rangle_\epsilon$ was bounded as $\epsilon \to 0$ we could have taken this limit and it would have been metric independent. One cannot expect this to be true. However, the divergences in $\langle \mathcal{O}\delta ||F_A||^2 \rangle_\epsilon$ for $\epsilon \to 0$ originate from a very definite type of contribution to the Feynman diagrams, and by considering how such divergences can originate, one can obtain results that are nearly as good as the naive results that would have held if there were no divergences. In explaining this, we will consider the basic case $\mathcal{O} = 1$.

It is convenient to consider only the connected Feynman diagrams and as is well known [36, 21, 29] the sum of those is just $\log\langle 1 \rangle_\epsilon$. Divergences in Feynman diagrammatic contributions to $\log\langle 1 \rangle_\epsilon$ and to

$$\delta \left(\log\langle 1 \rangle_\epsilon \right) = \frac{\epsilon \langle \delta ||F_A||^2 \rangle_\epsilon}{\langle 1 \rangle_\epsilon} \tag{10.2}$$

come from a region of integration in which all integration points are separated by distances of order ϵ. This means that the divergences can be expanded in terms of local differential geometric invariants – the metric, the curvature tensor, and its covariant derivatives. This expansion is analogous to the short time expansion of the heat kernel. The most general divergent terms are of the form

$$\log\langle 1 \rangle_\epsilon = \frac{c_1}{\epsilon^3} V + \frac{c_2}{\epsilon} R + \text{finite terms} \tag{10.3}$$

and

$$\frac{\langle \delta ||F_A||^2 \rangle_\epsilon}{\langle 1 \rangle_\epsilon} = \frac{c_1}{\epsilon^4} \delta V + \frac{c_2}{\epsilon^2} \delta R + \frac{c_3}{\epsilon} \delta C + \text{finite terms.} \tag{10.4}$$

Here c_1, c_2, and c_3 are constants (or more exactly functions of k only, which must be computed order by order in perturbation theory, but do not depend on the particular three manifold or metric). Also, V is the volume of M^3, R is the integral over M^3 of its scalar curvature, C is the Chern-Simons number associated with the Levi-Civita connection and δV, δR, δC are the variations of these quantities with respect to $g^{ij} \to g^{ij} + \delta g^{ij}$. The expansion (10.4) is determined by the following principles. (i) The terms on the right hand side must be closed one forms on the space of metrics (since the left hand side of the equation has this property.) (ii) The coefficients of these closed one forms must be local functionals of the metric. What has been written on the right hand side of equation (10.4) is the most general expression with these properties. The general principles do not determine c_1, c_2, and c_3, which from this point of view must simply be computed order by order in perturbation theory.

Equation (10.4) means that $\langle 1 \rangle_\epsilon$ does not converge as $\epsilon \to 0$ to a topological invariant. Indeed its variation (10.2) not only does not vanish as $\epsilon \to 0$; it diverges in this limit. If, however, we define[2]

$$\mathcal{W}_{renormalized} = \langle 1\rangle_{renormalized} \stackrel{\text{def}}{=} \exp\lim_{\epsilon\to 0}\left(\log\langle 1\rangle_\epsilon - \frac{c_1}{\epsilon^3}V - \frac{c_2}{\epsilon}R - c_3 C\right) \qquad (10.5)$$

then (10.3) shows that $\mathcal{W}_{renormalized}$ is finite while (10.1) and (10.4) shows that it is an invariant. Here we see where the framing of M^3 comes in — to define C we must first pick a trivialization of the tangent bundle and so the invariants that we have just produced depend on a choice of such a trivialization.

Notice that δC, in equation (10.4) does not depend on the choice of a framing, but C does. What is entering here is clearly a sort of local cohomology of the space of metrics. The local, closed one forms δV, δR appearing in (10.4) can be written as variations (exterior derivatives) of local functionals of the metric. But δC, though itself a local functional and a closed one form, cannot be written as the variation of a local functional. (If δC were itself not local, it could not arise in the intrinsic local evaluation of Feynman diagrams that leads to equation (10.4).)

Similarly, in the case of a non-empty link $\mathcal{X}$ we do not expect that the higher order Feynman diagrams will converge to link invariants, but instead we expect them to converge to something whose variation with respect to a deformation of $\mathcal{X}$ will be equal to some constant multiple of the variation of the total torsion of $\mathcal{X}$. (The torsion will enter just as the Chern-Simons number C entered in the above discussion.) The total torsion can then be subtracted out yielding link *invariants* at the price of having to introduce a framing for $\mathcal{X}$ — the total torsion can be defined only given such a framing. This agrees with the results of Witten and with the results in chapter 3.

Unfortunately, we were just *pretending* that the theory defined by $\mathcal{L}_{regularized}$ is finite. In fact, it is not. One can figure out how badly divergent the theories defined by $\mathcal{L}_{tot}$ and $\mathcal{L}_{regularized}$ are by taking a diagram with a specified number of vertices and arcs, measuring the total degree of singularity of the arcs and vertices, and subtracting the number of integrations that the vertices induce. The result, the so-called "superficial degree of divergence" Δ of a diagram with E_B external gauge lines, E_F external ghost lines and L internal loops is

$$\Delta(\mathcal{L}_{tot}) = 3 - E_B - \frac{1}{2}E_F \qquad ; \qquad \Delta(\mathcal{L}_{regularized}) = 4 - L - E_B - E_F. \qquad (10.6)$$

Clearly, the regularized theory is less divergent than the original one, but (10.6) shows that even in the regularized theory the diagrams with a small number of loops and external lines will be divergent and as these diagrams appear as subdiagrams in diagrams with higher complexity we cannot just ignore them. One can check that

[2]This is consistent with what is usually called renormalization - it just corresponds to adding $-\frac{c_1}{\epsilon^3}V - \frac{c_2}{\epsilon}R - c_3 C$ to the original Lagrangian as the limit $\epsilon \to 0$ is taken. In fact, the above paragraph can be summarized by saying that these three terms are the only possible local BRST invariant additions to the Lagrangian which are of the right dimension. Notice that all three terms depend on the metric alone and not on the fields, and therefore the n-point functions of the theory are not renormalized and thus no care needs to be taken of the renormalization of lower order diagrams when considering the renormalization of a fixed order in perturbation theory.

even if higher terms than $\epsilon||F_A||^2$ are added to $\mathcal{L}_{tot}$ and even when considering the reduction in the divergence that comes from gauge invariance[3] one loop diagrams with one, two , or three external legs will remain divergent in the resulting theory. Yet, we believe that the following is true:

Conjecture 1 *(Witten, [47]) The analysis in (10.3), (10.4), and (10.5) can be justified, and the resulting invariants $\mathcal{W}_{renormalized}$ coincide with the expansion in powers of $1/k$ of the results in [42, 43].*

One-loop diagrams in the Chern-Simons theory have been regularized using ζ-function regularization in [42, 7] and in chapter 5 of this thesis, and using Pauli-Villars regularization in [2]. In both these regularizations the 'shift in k' is observed consistently with the above conjecture.

[3] $Q\bar{c} = \phi$, and therefore $\langle \phi(x)\phi(y) \rangle = 0$. This together with the structure of the ϕB propagator proves that the amputated two-point function is given by $\star^L d^L$ of a $(1,1)$-form whose convergence properties are by one degree better. For a similar example, see e.g. [12, pp. 299-300].

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