FUNDAMENTAL CONCEPTS IN DIFFERENTIAL GEOMETRY
FALL 2000
EXERCISES HANDOUT # 11

1. Let G be a Lie group. Then G comes with a smooth multiplication map

$$\mu: G \times G \to G.$$

Show that the induced map

$$T_eG \times T_eG \simeq T_{(e\times e)}G \times G \xrightarrow{\mu_*} T_eG$$

coincides with the addition operation in T_eG

2. Let G be an abelian compact and connected Lie group. Let V be a real vector space and let $\Phi : V \to T_eG$ be a linear map. Show that there exists a Lie group homomorphism

$$\varphi: V \to G$$

which, at $0 \in V$, satisfies $\varphi_* = \Phi$.

3. (a) Let K be a subgroup of $\mathbb{R}^n$ which admits a neighborhood U if the identity such that $K \cap Y = \{e\}$. Prove that, up to a linear isomorphism, $K = \mathbb{Z}^k \leq \mathbb{Z}^n \leq \mathbb{R}^n$.

(b) Let G be a connected abelian compact Lie group. Show that $G \simeq \mathbb{R}^n/\mathbb{Z}^n$ for some n. We call G a *torus* (since $G \simeq (S^1)^{\times n}$). Conclude that G is generated by a single element. Can you say what $\mathrm{Aut}(G)$ is isomorphic to ?

Hint: Use the previous exercise with $V = T_eG$ and $\Phi = \mathrm{id} : V \to T_eG$. Then use (a) above.

4. Let G be a connected and compact Lie group. Show that there exists a maximal torus in G, i.e. a maximal connected abelian closed subgroup. Show that such a maximal torus T is self centralizing (i.e. $C_G(T) = T$) and that it has a finite index in its normalizer $N_G(T)$.

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