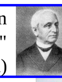
(u, v, and w knots) x (topology, combinatorics, low algebra, and high algebra)

Dror Bar−Natan, Penn State February 5 2009, http://www.math.toronto.edu/~drorbn/Talks/PSU−090205/

"God created the knots, all else in topology is the work of mortals." Leopold Kronecker (modified)



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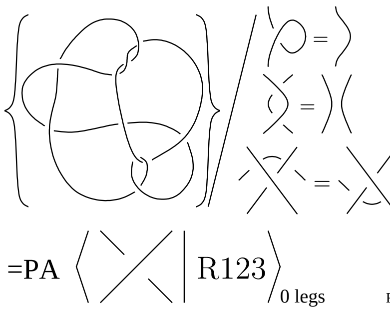
u−knots —1−1→ v−knots —onto→ w−knots 

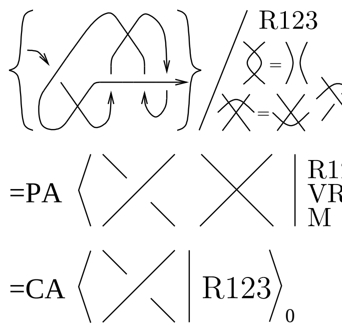
topology

u−knots are usual knots: 2

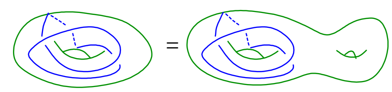


“Knots in $\mathbb{R}^3$”

v−knots are virtual knots: 3



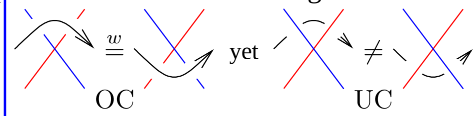
= Knots on surfaces, modulo stabilazation:

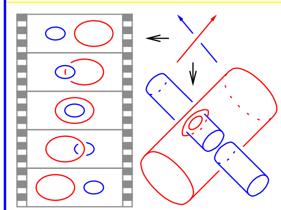


w is for welded, weakly v, and warmup:

4 {w−knots}={v−knots}/(OC)

where OC is Overcrossings Commute:



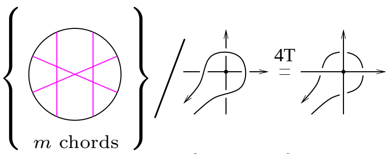


Related to "movies of flying rings" to knotted tubes in 4−space, and to "basis conjugating automorphisms of free groups".

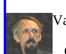
McCool Goldsmith Fenn Rimanyi Rourke Satoh Brendle Hatcher

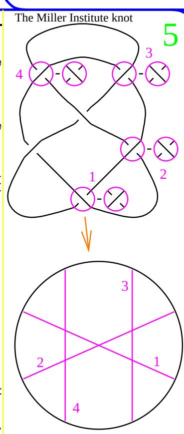
combinatorics

Extend any V : {u-knots} $\to A$ to “singular u-knots” using $V(\text{double point}) := V(\text{overcrossing}) - V(\text{undercrossing})$, and think “differentiation”.
Declare “V is of type m” iff $V^{(m+1)} \equiv 0$, think “polynomial of degree m”.
$W = V^{(m)}$ roughly determines V; $W \in \mathcal{A}_m^\star$ with



Need a “universal” Z : {u-knots} $\to \mathcal{A} = \bigoplus \mathcal{A}_m$.



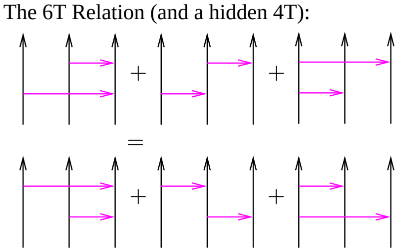


All the same, except 6

$V(\text{virtual double point}) := V(\text{overcrossing}) - V(\text{virtual crossing})$
$V(\text{virtual double point}) := V(\text{virtual crossing}) - V(\text{undercrossing})$
$\mathcal{A}^v := \{\text{“arrow diagrams”}\}/6T$

Need a Z : {v-knots} $\to \mathcal{A}^v$.

The 6T Relation (and a hidden 4T):

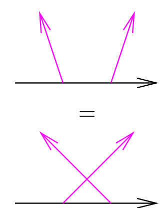


All the same, except 7

$\mathcal{A}^w := \mathcal{A}^v/TC$

Need a Z : {w-knots} $\to \mathcal{A}^w$.

"Tails Commute (TC)":



low algebra

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Similar

with metrized Lie algebras replacing arbitrary Lie algebras

So Far: Invariants for any $(\mathfrak{g}, R)$:

$$\{\text{knots}\} \xrightarrow[\text{high}]{Z} \mathcal{A} \xrightarrow[\text{low}]{T_\mathfrak{g}} \mathcal{U}(\mathfrak{g}) \xrightarrow{\text{Tr}_R} \mathbb{C}$$

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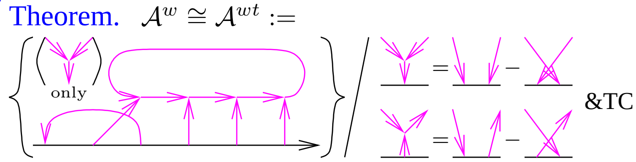
Similar

with Lie bi−algebras replacing arbitrary Lie algebras

m	1	2	3	4
dim $\mathcal{A}^v_m$	2	7	27	139
dim $\mathcal{L}ie_m$	2	7	27	≥ 118

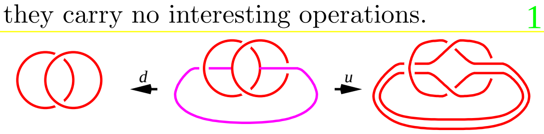
Theorem. $\mathcal{A}^w \cong \mathcal{A}^{wt} :=$ 8

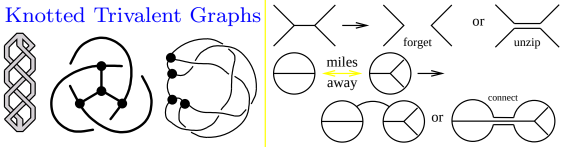


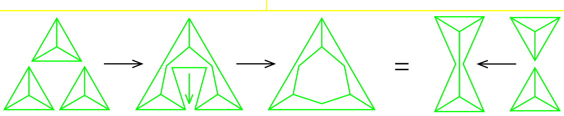
This screams, if you speak the language, LIE ALGEBRAS.
And indeed we have
Theorem. Given a finite dimensional Lie algebra $\mathfrak{g}$, there is $T : \mathcal{A}^w \to \mathcal{U}(I\mathfrak{g}) := \mathcal{U}(\mathfrak{g} \ltimes \mathfrak{g}^\star_{ab})$.

high algebra

Knots are the wrong objects to study in knot theory! They are not finitely generated and they carry no interesting operations. 11







Theorem (∼). A homomorphic Z is the same as a “Drinfel’d Associator”.

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Z is a Quantum Group?

More precisely, a homomorphic Z ought to be equivalent to the Etingof-Kazhdan theory of deformation quantization of Lie bialgebras.

Dror’s Dream: Straighten and fatten this column.

An Idle Question.
Is there physics in this column?

Switch to w-knotted trivalent tangles, 12

$$\text{wKTT} := CA\langle \text{overcrossing}, \text{undercrossing}, Y \rangle.$$

Theorem (∼). A homomorphic Z is equivalent to proving the Kashiwara-Vergne statement.

Statement (∼, KV, 1978) (proven Alekseev-Meinrenken, 2006). Convolutions of invariant functions on a group match with convolutions of invariant functions on its Lie algebra: for any finite dim. Lie group G with Lie algebra $\mathfrak{g}$,

$$(\text{Fun}(G)^{\text{Ad}\,G}, \star) \cong (\text{Fun}(\mathfrak{g})^{\text{Ad}\,G}, \star).$$

(Closely related to the “orbit method” of representation theory).