Dror Bar-Natan: Talks: Oberwolfach-0805:

Projectivization, Welded Knots and Alekseev-Torossian

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm.

Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y)=e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.
- The Alekseev-Torossian equations in $\mathcal{U}(\mathrm{sder}_n)$ and $\mathcal{U}(\mathrm{tder}_n)$.

sder $\leftrightarrow$ tree-level $\mathcal{A}$
tder $\leftrightarrow$ more

$$F\in\mathcal{U}(\mathrm{tder}_2);\quad F^{-1}e(x+y)F=e(x)e(y)\quad\Longleftrightarrow\quad F\in\mathrm{Sol}_0$$

$$\Phi=\Phi_F:=(F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23}\in\mathcal{U}(\mathrm{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}=\Phi^{12,3,4}\Phi^{1,2,34}\qquad\text{"the pentagon"}$$

$$t=\frac{1}{2}(y,x)\in\mathrm{sder}_2\text{ satisfies }4T\quad\text{and}\quad r=(y,0)\in\mathrm{tder}_2\text{ satisfies }6T$$

$$R:=e(r)\text{ satisfies Yang-Baxter: }R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$$

$$\text{also }R^{12,3}=R^{13}R^{23}\text{ and }F^{23}R^{1,23}(F^{23})^{-1}=R^{12}R^{13}$$

$$\tau(F):=RF^{21}e(-t)\text{ is an involution,}\Phi_{\tau(F)}=(\Phi_F^{321})^{-1}$$

$$\mathrm{Sol}_0^\tau:=\{F\colon\tau(F)=F\}\text{ is non-empty; for }F\in\mathrm{Sol}_0^\tau,$$

$$e(t^{13}+t^{23})=\Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and}\quad e(t^{12}+t^{13})=(\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$


Alekseev

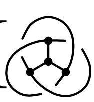

Torossian

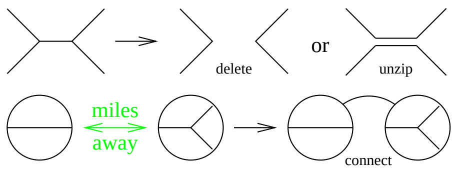
This is just a part of the Alekseev-Torossian work!

So What?

- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

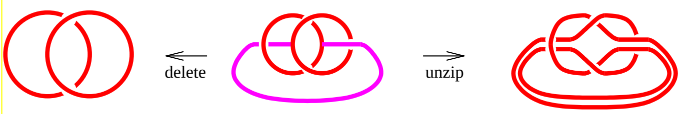
Knotted Trivalent Graphs

$\mathcal{O}(\triangle)=\{$  $,\dots\}$



Theorem. KTG is generated by the unknotted $\triangle$ and the Möbius band, with identifiable relations between them.

Theorem. $Z(\triangle)$ is equivalent to an associator Φ.

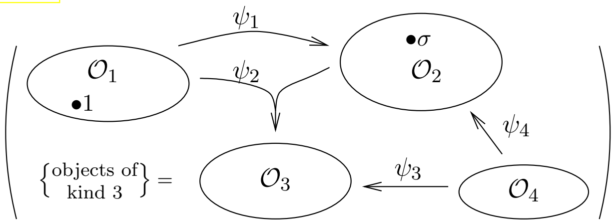


Algebraic Knot Theory

Theorem. {ribbon knots} $\sim\{u\gamma\colon\gamma\in\mathcal{O}(\circ\!\!\circ),\ d\gamma=\bigcirc\!\bigcirc\}$.

Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.

$\mathcal{O}=$ "An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining proj $\mathcal{O}$. The augmentation "ideal":

$$I=I_{\mathcal{O}}:=\left\{\begin{array}{l}\text{formal differences of ob-}\\\text{jects "of the same kind"}\end{array}\right\}.$$

$$\text{Then }I^n:=\left\{\begin{array}{l}\text{all outputs of algebraic}\\\text{expressions at least }n\text{ of}\\\text{whose inputs are in }I\end{array}\right\},\text{ and}$$

$$\mathrm{proj}\,\mathcal{O}:=\bigoplus_{n\geq 0}I^n/I^{n+1}.$$

- Has same kinds and operations, but different objects and axioms.

Knot Theory Anchors.

- $(\mathcal{O}/I^{n+1})^\star$ is "type n invariants".
- $(I^n/I^{n+1})^\star$ is "weight systems".
- $\mathrm{proj}\,\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".


Vassiliev


Goussarov

Warmup Examples.

- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$$1\wedge x=1,\quad x\wedge 1=x\wedge x=x,\qquad\text{(appetizers)}$$
$$(x\wedge y)\wedge z=(x\wedge z)\wedge(y\wedge z).\qquad\text{(main)}$$

$L:=\mathrm{proj}\,Q$ is a graded Lie algebra: set $\bar v:=(v-1)$ (these generate I!), feed $1+\bar x$, $1+\bar y$, $1+\bar z$ in (main), and collect the surviving terms of lowest degree:

$$(\bar x\wedge\bar y)\wedge\bar z=(\bar x\wedge\bar z)\wedge\bar y+\bar x\wedge(\bar y\wedge\bar z).$$

An Expansion is $Z\colon\mathcal{O}\to\mathrm{proj}\,\mathcal{O}$ s.t. $Z(I^n)\subset(\mathrm{proj}\,\mathcal{O})_{\geq n}$ and $Z_{I^n/I^{n+1}}=Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine $\mathrm{proj}\,\mathcal{O}$, but easy to guess a surjection $\rho:\mathcal{A}\to\mathrm{proj}\,\mathcal{O}$. So find $Z'\colon\mathcal{O}\to\mathcal{A}$ with $Z'(I^n)\subset\mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}}\circ\rho_n=Id_{\mathcal{A}_n}$:

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "Z'"] \arrow[dr, "Z"'] & \mathcal{A} \arrow[r, leftrightarrow] \arrow[d, "\rho"] & \mathcal{A}_n \\
& \mathrm{proj}\,\mathcal{O} \arrow[r, leftrightarrow] & (\mathrm{proj}\,\mathcal{O})_n \arrow[u, "Z'_{I^n/I^{n+1}}"']
\end{tikzcd}$$

Can you make this diagram less confusing?

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and $\mathrm{proj}\,\mathcal{O}$. They provide finite / combinatorial handles on global problems.


X-S. Lin

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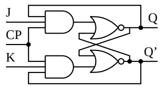
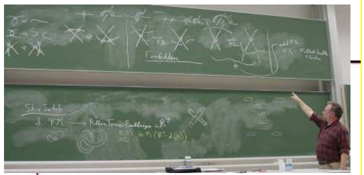
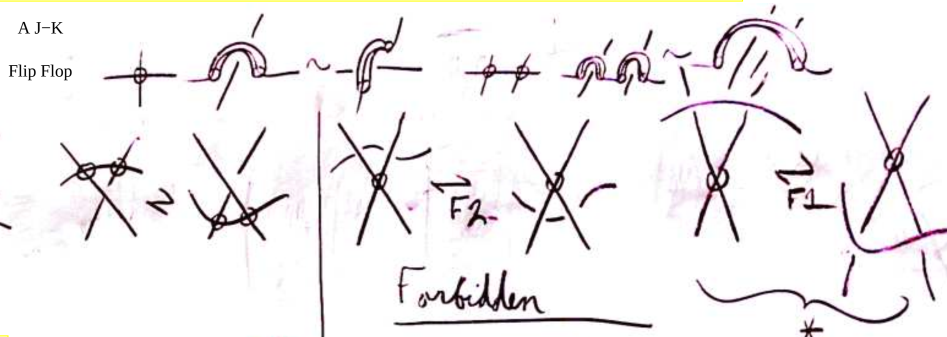
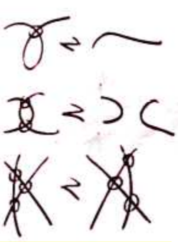


A J-K Flip Flop

Kauffman in Siegen



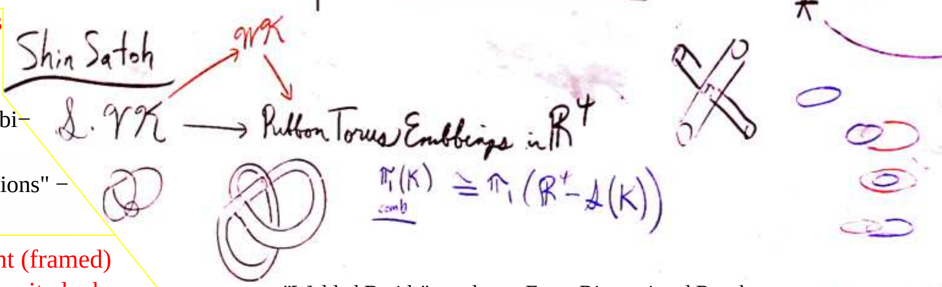




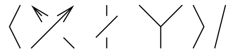
Circuit Algebras

* Have "circuits" with "ends"
* Can be wired arbitrarily.
* May have "relations" – de-Morgan, etc.

"Welded trivalent (framed) tangles" are a circuit algebra:

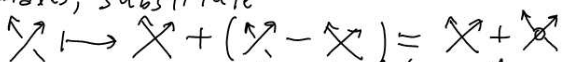


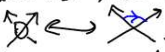
"Welded Braids" are due to Fenn, Rimanyi and Rourke

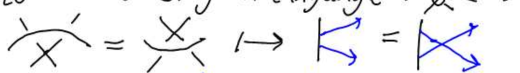
$\mathcal{WT} = \langle$  $\rangle$ / R123, R4 (for vertices), F1.

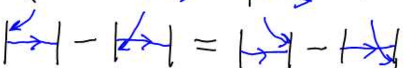
Further operations: delete, unzip.

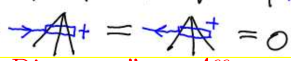
The "Chord Diagrams" — $\mathcal{A}^{wt}_n$. As we did for quandles, substitute



into the various moves, to get relations. Also switch to "arrow diagram language": . Get:

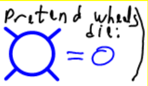




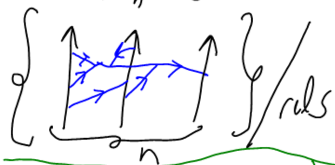


The "Jacobi Diagrams" — $\mathcal{A}^{cc}_n$.

Theorem. (95%) $\mathcal{A}^{wt}_n$ is $\mathcal{A}^{cc}_n$ is $\mathcal{U}(\mathrm{tder}_n)$.

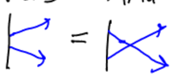
(Pretend wheels die:  = 0)

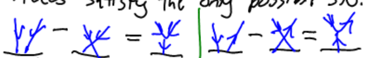
Here $\mathcal{A}^{cc}_n$ is



trivalent directed trees with only 2-in 1-out vertices 

In tensorland, this is "Co-commutative Lie-bialg"

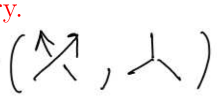
Rels: tails commute 

Heads satisfy the only possible STU:  + also IHX and vertex invariance

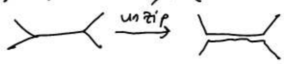
The Map α: $\mathcal{A}^{tree}_n \to \mathcal{A}^{cc}_n$: ——— $\mapsto$ →—— + ——←

Theorem. (90%) α is an injection on $\mathcal{A}^{tree}_n \cong \mathcal{U}(\mathrm{sder}_n)$. Furthermore, there is a simple charactarization of im α, so we can tell "an arrowless element" when we see it.

Partial Dictionary.

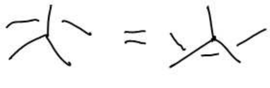
$(R, F) \longleftrightarrow ($  $)$ $\quad (r, t) \longleftrightarrow ($  $)$

$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \longleftrightarrow$ 

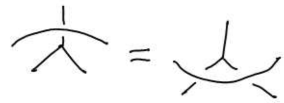
$FF^{-1} = I \longleftrightarrow$ 

$F^{-1}e(x+y)F = e(x)e(y)$

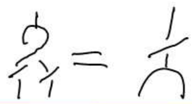
$\Updownarrow$

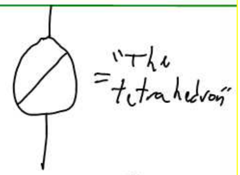
$F^{23}R^{1,23} = R^{12}R^{13}F^{23} \longleftrightarrow$ 

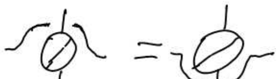
$R^{12,3} = R^{13}R^{23}$

$\Updownarrow$ $\longleftrightarrow$ 

$F^{12,3}R^{12,3} = R^{13}R^{23}F^{12,3}$ (unforbidding F1 makes this automatic)

$RF^{21}e(-t) = F \longleftrightarrow$ 

$\Phi = (F^{12,3})^{-1}(F^{1,2})^{-1}F^{2,13}F^{1,23} \longleftrightarrow$ 

Φ esdur $\longleftrightarrow$ 

The pentagon and the hexagons follow, with a minor twist, from the fact that we have an unzip-behaved invariant of KTG's.

The Main Theorem. (80%/0%) F's in Sol_0^τ are in a bijective correspondance with tree-level associators for ordinary paranthesized tangles (or ordinary knotted trivalent graphs) / with homomorphic expansions for knotted welded trivalent tangles.

Disclaimer. Orientations, rotation numbers, framings, the vertical direction and the cyclic symmetry of the vertex may still make everything uglier. I hope not.

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)

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This handout and further links are at http://www.math.toronto.edu/~drorbn/Talks/Oberwolfach-0805/