Projectivization, w-Knots, Kashiwara-Vergne and Alekseev-Torossian

The Categorification Speculative Paradigm.

- Every object in math is the Euler characteristic of a complex.
- Every operation in math lifts to an operation between complexes.
- Every identity in mathematics is true up to homotopy.

The Projectivization Tentative Speculative Paradigm. Projectivization?

- Every graded algebraic structure in mathematics is the projectivization of a plain ("global") one.
- Every equation written in a graded algebraic structure is an equation for a homomorphic expansion, or for an automorphism of such.

Graded Equations Examples

- $e(x+y)=e(x)e(y)$ in $\mathbb{Q}[[x,y]]$.
- The pentagon and hexagons in $\mathcal{A}(\uparrow_{3,4})$.
- The equations defining a QUEA, the work of Etingof and Kazhdan.

- The Alekseev-Torossian equations in $\mathcal{U}(\mathrm{sder}_n)$ and $\mathcal{U}(\mathrm{tder}_n)$.

sder $\leftrightarrow$ tree-level $\mathcal{A}$
tder $\leftrightarrow$ more

$$F\in\mathcal{U}(\mathrm{tder}_2);\quad F^{-1}e(x+y)F=e(x)e(y)\quad\Longleftrightarrow\quad F\in\mathrm{Sol}_0$$

$$\Phi=\Phi_F:=(F^{12,3})^{-1}(F^{1,2})^{-1}F^{23}F^{1,23}\in\mathcal{U}(\mathrm{sder}_3)$$

$$\Phi^{1,2,3}\Phi^{1,23,4}\Phi^{2,3,4}=\Phi^{12,3,4}\Phi^{1,2,34}\qquad\text{"the pentagon"}$$

$$t=\frac{1}{2}(y,x)\in\mathrm{sder}_2 \text{ satisfies } 4T \quad\text{and}\quad r=(y,0)\in\mathrm{tder}_2 \text{ satisfies } 6T$$

$$R:=e(r) \text{ satisfies Yang-Baxter: } R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12}$$

$$\text{also } R^{12,3}=R^{13}R^{23} \text{ and } F^{23}R^{1,23}(F^{23})^{-1}=R^{12}R^{13}$$

$$\tau(F):=RF^{21}e(-t) \text{ is an involution}, \Phi_{\tau(F)}=(\Phi_F^{321})^{-1}$$

$$\mathrm{Sol}_0^\tau:=\{F\colon \tau(F)=F\} \text{ is non-empty; for } F\in\mathrm{Sol}_0^\tau,$$

$$e(t^{13}+t^{23})=\Phi^{213}e(t^{13})(\Phi^{231})^{-1}e(t^{23})\Phi^{321}$$

$$\text{and}\quad e(t^{12}+t^{13})=(\Phi^{132})^{-1}e(t^{13})\Phi^{312}e(t^{12})\Phi$$

Alekseev

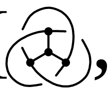
Torossian

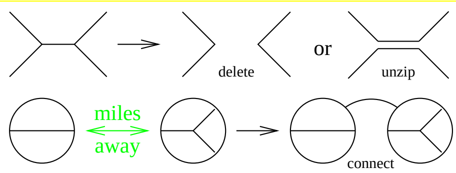
This is just a part of the Alekseev-Torossian work!

So What?

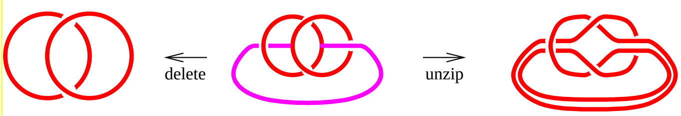
- Related to the Kashiwara-Vergne Conjecture!
- Will likely lead to an explicit tree-level associator, a linear equation away from a 1-loop equation, two linear equations away from a 2-loop associator, etc.!
- A baby version of the QUEA equations; we may be on the right tracks!

Knotted Trivalent Graphs

$\mathcal{O}(\triangle)=$ { , ... }



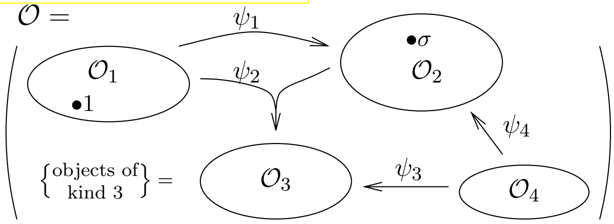
Theorem. KTG is generated by the unknotted $\triangle$ and the Möbius band, with identifiable relations between them.
Theorem. $Z(\triangle)$ is equivalent to an associator Φ.



Algebraic Knot Theory

Theorem. {ribbon knots} $\sim\{u\gamma\colon\gamma\in\mathcal{O}(\circ\!\!\circ),\ d\gamma=\bigcirc\bigcirc\}$.
Hence an expansion for KTG may tell us about ribbon knots, knots of genus 5, boundary links, etc.

"An Algebraic Structure"



- Has kinds, objects, operations, and maybe constants.
- Perhaps subject to some axioms.
- We always allow formal linear combinations.

Defining proj $\mathcal{O}$. The augmentation "ideal":

$$I=I_{\mathcal{O}}:=\left\{\begin{matrix}\text{formal differences of ob-}\\ \text{jects "of the same kind"}\end{matrix}\right\}.$$

$$\text{Then } I^n:=\left\{\begin{matrix}\text{all outputs of algebraic}\\ \text{expressions at least } n \text{ of}\\ \text{whose inputs are in } I\end{matrix}\right\}, \text{ and}$$

$$\operatorname{proj}\mathcal{O}:=\bigoplus_{n\geq 0}I^n/I^{n+1}\quad\left(\begin{matrix}\text{has same kinds and opera-}\\ \text{tions, but different objects}\\ \text{and axioms}\end{matrix}\right).$$

Knot Theory Anchors.

Vassiliev

Goussarov

- $(\mathcal{O}/I^{n+1})^\star$ is "type n invariants".
- $(I^n/I^{n+1})^\star$ is "weight systems".
- proj $\mathcal{O}$ is $\mathcal{A}$, "chord diagrams".

Warmup Examples.

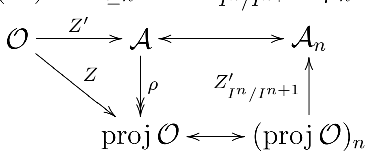
- The projectivization of a group is a graded associative algebra.
- A quandle: a set Q with a binary op $\wedge$ s.t.

$$1\wedge x=1,\quad x\wedge 1=x\wedge x=x,\qquad\text{(appetizers)}$$
$$(x\wedge y)\wedge z=(x\wedge z)\wedge(y\wedge z).\qquad\text{(main)}$$

proj Q is a graded Lie algebra: set $\bar v:=(v-1)$ (these generate I!), feed $1+\bar x$, $1+\bar y$, $1+\bar z$ in (main), collect the surviving terms of lowest degree:

$$(\bar x\wedge\bar y)\wedge\bar z=(\bar x\wedge\bar z)\wedge\bar y+\bar x\wedge(\bar y\wedge\bar z).$$

An Expansion is $Z\colon\mathcal{O}\to\operatorname{proj}\mathcal{O}$ s.t. $Z(I^n)\subset(\operatorname{proj}\mathcal{O})_{\geq n}$ and $Z_{I^n/I^{n+1}}=Id_{I^n/I^{n+1}}$ (A "universal finite type invariant"). In practice, it is hard to determine proj $\mathcal{O}$, but easy to guess a surjection $\rho:\mathcal{A}\to\operatorname{proj}\mathcal{O}$. So find $Z'\colon\mathcal{O}\to\mathcal{A}$ with $Z'(I^n)\subset\mathcal{A}_{\geq n}$ and $Z'_{I^n/I^{n+1}}\circ\rho_n=Id_{\mathcal{A}_n}$:



Can you make this diagram less confusing?

X-S. Lin

Homomorphic Expansions are expansions that intertwine the algebraic structure on $\mathcal{O}$ and proj $\mathcal{O}$. They provide finite / combinatorial handles on global problems.

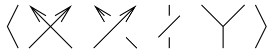
The Key Point. If $\mathcal{O}$ is finitely presented, finding a homomorphic expansion is solving finitely many equations with finitely many unknowns, in some graded spaces.

We Mean Business

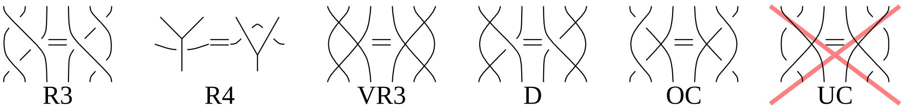
Trivalent (framed) w−tangles:

further operations: delete, unzip.

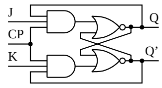
$wTT = CA\langle$  $\rangle$ / R123, R4 (for vertices), F, OC.

$= PA\langle$  $\rangle$ / R1234, F, VR1234, D, OC.

(=tangles in thick surfaces, modulo stabilization)

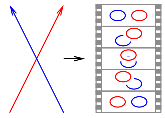


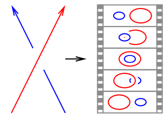
Circuit Algebras



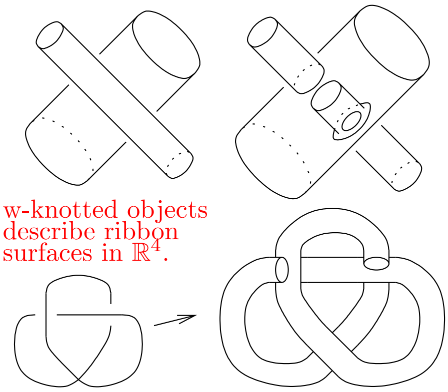
* Have "circuits" with "ends"
* Can be wired arbitrarily.
* May have "relations" − de−Morgan, etc.

w−braids describe flying rings:



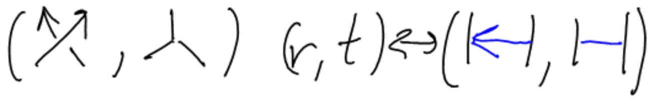


w-knotted objects describe ribbon surfaces in $\mathbb{R}^4$.

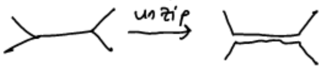


(and π_1 is preserved)

Partial Dictionary.

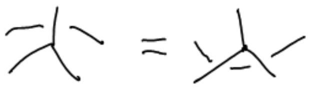
$(R,F) \longleftrightarrow ($  $)$ $(r,t) \longleftrightarrow ($ ... $)$

$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \longleftrightarrow$ 

$FF^! = I \longleftrightarrow$ 

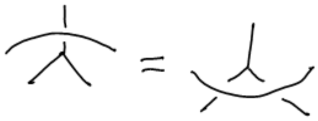
$F^{-1}e(x+y)F = e(x)e(y)$

⇕ (1)

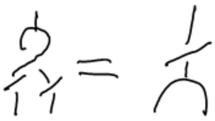
$F^{23}R^{1,23} = R^{12}R^{13}F^{23} \longleftrightarrow$ 

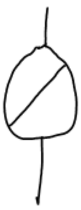
$R^{12,3} = R^{13}R^{23}$

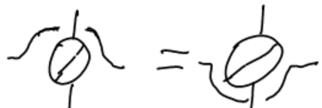
⇕

$F^{12,3}R^{12,3} = R^{13}R^{23}F^{12,3} \longleftrightarrow$ 

(unforbidding FI makes this automatic)

$RF^{21}e(-t) = F \longleftrightarrow$ 

$\Phi = (F^{12,3})^{-1}(F^{1,2})^{-1}F^{2,13}F^{1,23} \longleftrightarrow$  = "The tetrahedron"

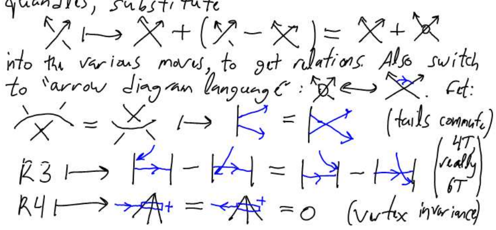
$\Phi \in \mathrm{sder} \longleftrightarrow$ 

The pentagon and the hexagons follow, with a minor twist, from the fact that we have an unzip-behaved invariant of KTG's.

For the Experienced

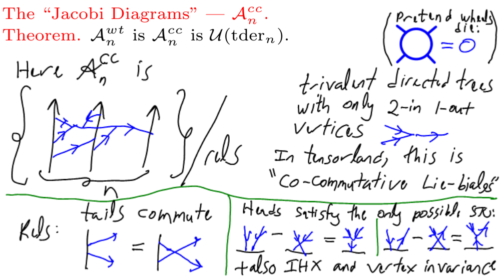
(and sharp-eyed)

The "Chord Diagrams" — $\mathcal{A}_n^{wt}$. As we did for quandles, substitute



The "Jacobi Diagrams" — $\mathcal{A}_n^{cc}$.

Theorem. $\mathcal{A}_n^{wt}$ is $\mathcal{A}_n^{cc}$ is $\mathcal{U}(\mathrm{tder}_n)$.



The Map α: $\mathcal{A}_n^{tree} \to \mathcal{A}_n^{cc}$: 

Theorem. α is an injection on $\mathcal{A}_n^{tree} \cong \mathcal{U}(\mathrm{sder}_n)$. Furthermore, there is a simple characterization of im α, so we can tell "an arrowless element" when we see it.

The Main Theorem. (approximate, false as stated) F's in Sol_0^τ are in a bijective correspondance with tree-level associators for ordinary paranthesized tangles (or ordinary knotted trivalent graphs) / with homomorphic expansions for trivalent w-tangles / with solutions of the Kashiwara-Vergne problem.

Extra. Restricted to knots, we get precisely the Alexander polynomial.

Disclaimer. Orientations, rotation numbers, framings, the vertical direction and the cyclic symmetry of the vertex may still make everything uglier. I hope not.

"God created the knots, all else in topology is the work of mortals"

Leopold Kronecker (paraphrased)

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