Workshop "GRT, MZVs and associators" Les Diablerets, 21 August 2015
9:30 – 11:30

"Algebraic aspects of the Goldman–Turaev Lie bialgebra"

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joint work with Yusuke Kuno (Tsuda College)

survey paper: arXiv: 1304.1885 (to appear in: Handbook of Teichmüller Theory vol. 5.)

◎ Outline of this talk

- I. Goldman bracket ---- intersections of 2 curves on a surface
- II. Turaev cobracket --- self-intersections of a single curve

I. Goldman bracket

- tensorial description of the Goldman Lie algebra

  → positive genus with connected boundary
  symplectic derivation algebra

  → genus 0 with $n+1$ ∂-components

  - Kontsevich's "Associative" worl(l)d
  - Morita's Lie algebra
    $\fallingdotseq$ Kontsevich's "Lie" worl(l)d

  special derivation algebra of { free associative algebra / free Lie algebra }

$\longrightarrow$ a natural embedding τ of the Torelli group into a completion of the Goldman Lie algebra

II. Turaev cobracket δ

- Image $\tau \subset \operatorname{Ker} \delta$
- tensorial description of the Turaev cobracket ?

  --- partial results

  - (Massuyeau-Turaev, Kuno-K.) lowest degree term = Schedler's cobracket
  - (K.) lowest degree term of a regular homotopy version of the Turaev cobracket

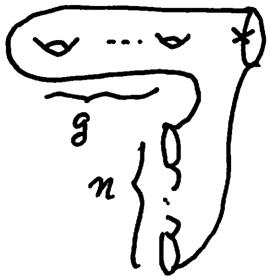
    $\longrightarrow$ Enomoto-Satoh trace (positive genus with conn. ∂)

    $\longrightarrow$ divergence cocycle in the Kashiwara-Vergne problem (genus 0)
  - (Fukuhara-Kuno-K.) a relation with the Bernoulli numbers

I. Goldman bracket

S : compact connected oriented C^∞ surface with $\partial S \neq \phi$

$\Longrightarrow$ (Classification Theorem) $\exists g, \exists n \geq 0 \quad S \cong \Sigma_{g,n+1} =$



$* \in \partial S$

$\pi := \pi_1(S, *) \cong F_{2g+n}$

free group of rank $2g+n$

$\mathcal{M}(S) \stackrel{\text{def}}{=} \pi_0 \mathrm{Diff}^+(S, 1_{\partial S} \text{ on } \partial S)$

$= \{ \varphi : S \to S$: (ori-pres) diffeomorphism, $\varphi|_{\partial S} = 1_{\partial S} \}$ / isotopy fixing ∂S pointwise

mapping class group

$\partial S = \coprod_{k=0}^{n} \partial_k S$, Choose $*_k \in \partial_k S$, $0 \leq k \leq n$, $* = *_0 \in \partial_0 S$

$E := \{*_k\}_{k=0}^{n} \subset \partial S$

$(\Pi S)(*_k, *_\ell) = \pi_1(S, *_k, *_\ell) := \{ \ell : ([0,1], 0, 1) \to (S, *_k, *_\ell)$; continuous map $\} / \simeq$ rel ∂

$\Pi S|_E$: restriction of the fundamental groupoid ΠS to the set E

(object $*_k \in E$, $0 \leq k \leq \ell$, morphisms $\Pi S(*_k, *_\ell)$, $0 \leq k, \ell \leq n$)

(Dehn-Nielsen type Theorem

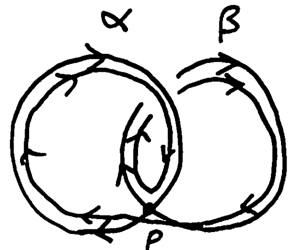
$DN : \mathcal{M}(S) \longrightarrow \mathrm{Aut}(\Pi S|_E)$, $\varphi \mapsto \varphi_*$, is injective

$\hat{\pi} = \hat{\pi}(S) := \pi/\text{conjugate} = \{\text{free loops on } S\}/\text{free homotopy}$

$p \in S$, $\quad |\ | : \pi_1(S,p) \longrightarrow \hat{\pi} = \hat{\pi}(S)$ forgetful map of the basepoint p

Goldman bracket

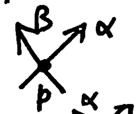
$\alpha, \beta \in \hat{\pi}$ (Choose their representatives) in general position.

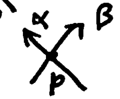


$\varepsilon_p = -1$

$$[\alpha, \beta] \overset{\mathrm{def}}{=} \sum_{p \in \alpha \cap \beta} \varepsilon_p(\alpha, \beta)\, |\alpha_p \beta_p| \in \mathbb{Z}\hat{\pi}$$

where $\varepsilon_p(\alpha,\beta) = \begin{cases} +1 & \text{if} \\ -1 & \text{if} \end{cases}$





local intersection number

α_p (resp. β_p) $\in \pi_1(S,p)$ based loop along α (resp. β)

Goldman

(1) $[\ ,\ ]$: well-defined

(2) $(\mathbb{Z}\hat{\pi}, [\ ,\ ])$: Lie algebra ($\rightsquigarrow$ the Goldman Lie algebra of S)

Backgrounds

(1) (Wolpert) Weil-Petersson geometry

(2) (Atiyah-Bott-Goldman) Poisson structure on the moduli of flat bundles

Generalization to higher dimensions

(Chas-Sullivan) String Topology

$\mathbb{1} := |1| \in \hat{\pi}$ constant loop , $\mathbb{1} \in \mathrm{Center}(\mathbb{Z}\hat{\pi})$

$\mathbb{Z}\hat{\pi}' := \mathbb{Z}\hat{\pi} / \mathbb{Z}\mathbb{1}$ quotient Lie algebra

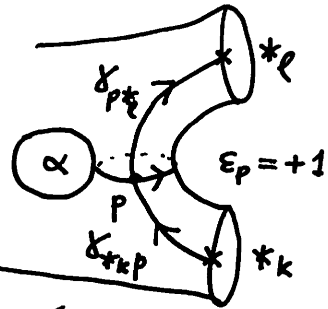
$|\ |' : \mathbb{Z}\pi \xrightarrow{|\ |} \mathbb{Z}\hat{\pi} \xrightarrow{\text{quotient}} \mathbb{Z}\hat{\pi}/\mathbb{Z}\mathbb{1} = \mathbb{Z}\hat{\pi}'$

Action of $\mathbb{Z}\hat{\pi}'$ on the groupoid $\Pi S|_E$

$E = \{*_k\}_{k=0}^{\infty} \subset \partial S$, $*_k \in \partial_k S$, $0 \le k \le n$,

$\left.\begin{array}{l} \alpha \in \hat{\pi}(S) \text{ free loop} \\ \gamma \in \Pi S(*_k, *_\ell) \text{ path from } *_k \text{ to } *_\ell,\ 0 \le k, \ell \le n \end{array}\right\}$ in general position

$$\sigma(\alpha)(\gamma) \overset{\text{def}}{=} \sum_{p \in \alpha \cap \gamma} \varepsilon_p(\alpha, \gamma)\, \gamma_{*_k p}\, \alpha_p\, \gamma_{p *_\ell} \in \mathbb{Z}\Pi S(*_k, *_\ell)$$



Kuno-K.

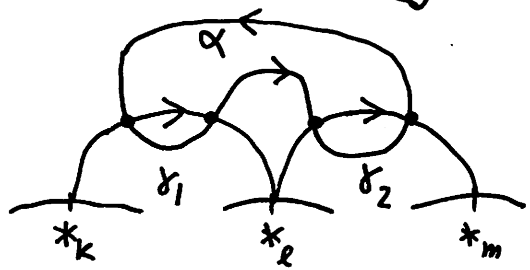
(1) σ : well-defined

(2) $\sigma(\alpha)$: derivation of $\mathbb{Z}\Pi S|_E$

$$\sigma(\alpha)(\gamma_1\gamma_2) = \sigma(\alpha)(\gamma_1)\,\gamma_2 + \gamma_1\,\sigma(\alpha)(\gamma_2)$$

(3) $\sigma : \mathbb{Z}\hat{\pi}' \longrightarrow \mathrm{Der}(\mathbb{Z}\Pi S|_E)$

Lie algebra homomorphism



<u>Completions</u> / $\mathbb{Q}$ (↦ can be replaced by any field of char. 0)

$\varepsilon : \mathbb{Q}\pi \to \mathbb{Q}$, $\sum_{x\in\pi} a_x x \mapsto \sum_{x\in\pi} a_x$ augmentation map

$I\pi := \mathrm{Ker}(\varepsilon : \mathbb{Q}\pi \to \mathbb{Q})$ augmentation ideal

$\widehat{\mathbb{Q}\pi} := \varprojlim_{p\to\infty} \mathbb{Q}\pi / (I\pi)^p$ completed group ring

(ex) $\forall x \in \pi$, $\frac{1}{2}(\log x)^2 \in \widehat{\mathbb{Q}\pi}$)

$p \geq 0$, $\mathbb{Q}\hat{\pi}(p) := \mathbb{Q}\mathbb{1} + |(I\pi)^p| \subset \mathbb{Q}\hat{\pi}$ linear subspace

$\forall p_1, \forall p_2 \geq 1$ $\sigma(\mathbb{Q}\hat{\pi}(p_1))(\gamma_k (I\pi)^{p_2} \gamma_\ell^{-1}) \subset \gamma_k (I\pi)^{p_1+p_2-2} \gamma_\ell^{-1}$ $\left(\begin{array}{l} \gamma_k \in \Pi S(*_k, *_0) \\ \gamma_\ell \in \Pi S(*_\ell, *_0) \end{array} \right)$

$[\mathbb{Q}\hat{\pi}(p_1), \mathbb{Q}\hat{\pi}(p_2)] \subset \mathbb{Q}\hat{\pi}(p_1+p_2-2)$

$\widehat{\mathbb{Q}\hat{\pi}} := \varprojlim_{p\to\infty} \mathbb{Q}\hat{\pi} / \mathbb{Q}\hat{\pi}(p)$ completed Goldman Lie algebra

(ex) $\forall x \in \pi$, $\frac{1}{2}(\log |x|)^2 := |\frac{1}{2}(\log x)^2| \in \widehat{\mathbb{Q}\hat{\pi}}$)

$(\widehat{\mathbb{Q}\Pi S}|_E)(*_k, *_\ell) := \varprojlim_{p\to\infty} \mathbb{Q}\Pi S(*_k, *_\ell) / \gamma_k (I\pi)^p \gamma_\ell^{-1}$ (independent of the choice of γ_k and γ_ℓ)

$\sigma : \widehat{\mathbb{Q}\hat{\pi}} \to \mathrm{Der}(\widehat{\mathbb{Q}\Pi S}|_E)$ (continuous) Lie algebra homomorphism

$$\mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E) := \{ D \in \mathrm{Der}(\widehat{\mathbb{Q}\Pi S}|_E) : \left(\begin{smallmatrix}\text{continuous}\\ \text{derivation}\end{smallmatrix}\right),\ 0 \le {}^\forall k \le n,\ D(\partial_k S) = 0 \}$$
$$\supset \sigma(\widehat{\mathbb{Q}\hat{\pi}})$$

Theorem (Kuno-K.) S: compact connected oriented C^∞ surface with $\partial S \neq \phi$

$\Rightarrow$ $\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E)$ topological isomorphism of Lie algebras

(---- an immediate consequence of a tensorial description of the Goldman Lie algebra)

(ex) ${}^\forall \alpha \in \hat{\pi}(S)$, ${}^\forall \gamma \in \Pi S(*_k, *_l)$ $\sigma(\log \alpha)(\gamma) = ([\alpha]\cdot[\gamma])\gamma$)

$\widehat{DN} : \mathcal{M}(S) \to \mathrm{Aut}(\widehat{\mathbb{Q}\Pi S}|_E)$, $\varphi \mapsto \varphi_*$,

$\mathcal{M}(S)^0 := \{ \varphi \in \mathcal{M}(S) ;\ \log \widehat{DN}(\varphi) := \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} (\widehat{DN}(\varphi) - 1)^m$ converges $\}$

subset of $\mathcal{M}(S)$

(ex) • Dehn twist $\in \mathcal{M}(S)^0$, • Torelli group $\subset \mathcal{M}(S)^0$)

$\log \circ \widehat{DN} : \mathcal{M}(S)^0 \to \mathrm{Der}_\partial(\widehat{\mathbb{Q}\Pi S}|_E)$ injective

$\tau := \sigma^{-1} \circ \log \circ \widehat{DN} : \mathcal{M}(S)^0 \to \widehat{\mathbb{Q}\hat{\pi}}(S)$ geometric Johnson homomorphism

Theorem ($\Sigma_{g,1}$ Kuno-K.; general Kuno-K., Massuyeau-Turaev) ${}^\forall C \subset S$ simple closed curve

τ(right-handed Dehn twist along C) $= \frac{1}{2}(\log C)^2 \in \widehat{\mathbb{Q}\hat{\pi}}(S)$

(Rmk $(\log C)^2 \neq (\log C)$"$\times$"$(\log C)$)

ex) $S = \Sigma_{g,1}$, $\mathcal{I}_{g,1} := \mathrm{Ker}(\mathcal{M}(\Sigma_{g,1}) \to \mathrm{Aut}(H_1(\Sigma_{g,1};\mathbb{Z})))$ Torelli group

$\tau|_{\mathcal{I}_{g,1}}$: equivalent to Massuyeau's Johnson map

$\mathrm{gr}(\tau|_{\mathcal{I}_{g,1}})$: classical Johnson homomorphism

Tensorial description

$\pi = \pi_1(S,*)$ free group of finite rank

$H := (\pi/[\pi,\pi]) \otimes_{\mathbb{Z}} \mathbb{Q} = H_1(S;\mathbb{Q})$, $\gamma \in \pi \mapsto [\gamma] := (\gamma \bmod [\pi,\pi]) \otimes 1 \in H$

$\widehat{T} = \widehat{T}(H) := \prod_{m=0}^{\infty} H^{\otimes m}$ completed tensor algebra

$p \geq 1$, $\widehat{T}_{\geq p} := \prod_{m \geq p} H^{\otimes m} \subset \widehat{T}$ two-sided ideal $\dashrightarrow$ topology of $\widehat{T}$

$\Delta : \widehat{T} \to \widehat{T} \hat{\otimes} \widehat{T}$ coproduct, $X \in H \mapsto X \hat{\otimes} 1 + 1 \hat{\otimes} X$

$\iota : \widehat{T} \to \widehat{T}$ antipode $\iota(X_1 X_2 \cdots X_m) = (-1)^m X_m \cdots X_2 X_1$, $X_j \in H$

$\widehat{T}$: complete Hopf algebra

$\widehat{\mathcal{L}} = \widehat{\mathcal{L}}(H) := \{u \in \widehat{T} : \Delta u = u \hat{\otimes} 1 + 1 \hat{\otimes} u\}$ Lie-like element

free Lie algebra over H

- <u>cyclic symmetrizer</u>

  $N : \hat{T} \to \hat{T}$ continuous linear map

  $N|_{H^{\otimes 0}} := 0$

  $N(X_1 \cdots X_m) := \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1} \quad (X_j \in H)$

  $N(\hat{T}) = \prod_{m=1}^{\infty} (H^{\otimes m})^{\mathbb{Z}/m} \subset \hat{T}_{\geq 1}$ cyclic invariants

- <u>Definition</u> $\theta : \pi \to \hat{T}$ group-like expansion

  $\overset{\text{def}}{\Longleftrightarrow}$ 1) (multiplicative) $\forall \gamma, \forall \delta \in \pi \quad \theta(\gamma\delta) = \theta(\gamma)\theta(\delta)$

  2) $\forall \gamma \in \pi \quad \theta(\gamma) \equiv 1 + [\gamma] \mod \hat{T}_{\geq 2}$

  3) (group-like) $\forall \gamma \in \pi \quad \Delta\theta(\gamma) = \theta(\gamma) \hat{\otimes} \theta(\gamma)$

  $\theta : \widehat{\mathbb{Q}\pi} \xrightarrow{\cong} \hat{T}, \quad \sum_{x \in \pi} a_x x \mapsto \sum_{x \in \pi} a_x \theta(x),$

  isom. of complete Hopf algebras

  $N\theta : \mathbb{Q}\hat{\hat{\pi}} \xrightarrow{\cong} N(\hat{T}), \quad |\gamma| \in \hat{\pi} \mapsto N\theta(\gamma)$

  isom. of (topological) $\mathbb{Q}$-vector spaces

$\boxed{\Sigma_{g,1}}$

$$H = H_1(\Sigma_{g,1};\mathbb{Q}) \xrightarrow[\text{duality}]{\text{Poincaré}} H^1(\Sigma_{g,1};\mathbb{Q}) = H^*$$

$$X \longmapsto (Y \mapsto X\cdot Y) \quad , \quad X\cdot Y \in \mathbb{Q} \text{ intersection number}$$

$$\omega := \sum_{i=1}^{g} A_i B_i - B_i A_i \in H^{\otimes 2} < \hat{T} \quad \text{symplectic form}$$

independent of the choice of a symplectic basis $\{A_i, B_i\}_{i=1}^{g} < H$

$$\begin{array}{ccccccc} \mathrm{Der}(\hat{T}) & \underset{\substack{\text{restriction}\\ \text{to } H}}{\cong} & \mathrm{Hom}(H,\hat{T}) = H^*\otimes\hat{T} & \overset{\text{P.d.}}{=} & H\otimes\hat{T} & = & \hat{T}_{\geq 1} \\ \cup & & \circlearrowright & & & & \cup \end{array}$$

$$\mathrm{Der}_\omega(\hat{T}) := \{D : \overset{\text{(continuous)}}{\text{derivation}} \text{ of } \hat{T} : D\omega = 0\} \cong N(\hat{T})$$

$\mathrm{Der}_\omega(\hat{T}) = N(\hat{T}) \supset N(\hat{T}_{\geq 2})$: degree completion of Kontsevich's "associative" worl(l)d

$\cup$

$\mathrm{Der}_\omega(\hat{\mathcal{L}}) = \{D \in \mathrm{Der}_\omega(\hat{T}) : \Delta\circ D = (D\hat{\otimes}1 + 1\hat{\otimes}D)\circ\Delta\} = N(H\otimes\hat{\mathcal{L}})$

$\cup$ degree completion of Kontsevich's "Lie" worl(l)d

$N(H\otimes\hat{\mathcal{L}}_{\geq 2})$: degree completion of Morita's Lie algebra $\mathfrak{h}_{g,1}$

"Morita's target of the Johnson homomorphisms

$\zeta \in \pi = \pi_1(\Sigma_{g,1}, *)$ negative boundary loop

<u>Definition</u> (Massuyeau)

$\theta : \pi = \pi_1(\Sigma_{g,1}, *) \to \hat{T} = \hat{T}(H_1(\Sigma_{g,1} : \mathbb{Q}))$ symplectic expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\theta : \pi \to \hat{T}$ group-like expansion

2) (symplectic) $\theta(\zeta) = e^{\omega} \left(= \sum_{m=0}^{\infty} \frac{1}{m!} \omega^m \right) \in \hat{T}$

<u>examples</u> 1) (K.) analytic construction $/\mathbb{R}$

2) (Massuyeau) Lê-Murakami-Ohtsuki functor $/\mathbb{Q}$

3) (Kuno) combinatorial construction $/\mathbb{Q}$

<u>Theorem</u> (Kuno-K.) $S = \Sigma_{g,1}$, θ : symplectic expansion

$\Longrightarrow$ (1) $-N\theta : \mathbb{Q}\hat{\pi} \xrightarrow{\cong} N(\hat{T}) = \mathrm{Der}_{\omega}(\hat{T})$ (top) isom of Lie algebras

(2)

$$\begin{tikzcd}
\widehat{\mathbb{Q}\hat{\pi}} \times \widehat{\mathbb{Q}\pi} \arrow[r, "\sigma"] \arrow[d, "\cong", "(-N\theta)\times\theta"'] \arrow[dr, phantom, "\circlearrowleft"] & \widehat{\mathbb{Q}\pi} \arrow[d, "\theta", "\cong"'] \\
\mathrm{Der}_{\omega}(\hat{T}) \times \hat{T} \arrow[r, "\text{derivation}"'] & \hat{T}
\end{tikzcd}$$

$\boxed{\text{general case}}$ $\Sigma_{g,n+1}$, $g, n \geq 0$,

- <u>Massuyeau–Turaev</u> : tensorial description of the homotopy intersection form
- <u>Kuno–K.</u> $\overline{S} := S \underset{\partial S}{\cup} ((n+1)\text{ disks}) \cong \Sigma_g =$ 

  Fix a section s of the inclusion homom. $\mathrm{incl}_* : H_1(S;\mathbb{Q}) \twoheadrightarrow H_1(\overline{S};\mathbb{Q})$

(<u>remark</u> If $S = \Sigma_{g,1}$ or $\Sigma_{0,n+1}$, then s is unique)

$N(\hat{T})_s = (N(\hat{T}), [\ ,\ ]_s)$: Lie algebra structure on $N(\hat{T})$ induced by the section s

<u>Theorem</u> (Kuno–K.; can be deduced from Massuyeau–Turaev's description)

$\forall g, \forall n \geq 0$ $S = \Sigma_{g,n+1}$, s: a section of $\mathrm{incl}_* : H_1(S;\mathbb{Q}) \twoheadrightarrow H_1(\overline{S};\mathbb{Q})$

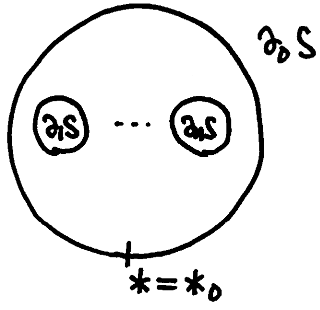
θ : "symplectic expansion" of $\pi S|_E$ compatible with s

$\Rightarrow -N\theta : \widehat{\mathbb{Q}\hat{\pi}}(S) \xrightarrow{\cong} N(\hat{T})_s$ (top) isom. of Lie algebras

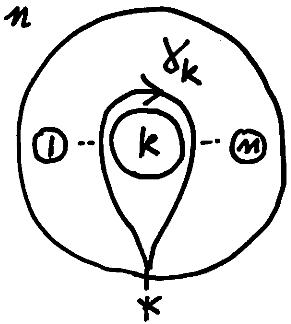
($\leftarrow\dashrightarrow$ <u>Thm</u>: $\sigma : \widehat{\mathbb{Q}\hat{\pi}}(S) \cong \mathrm{Der}_\partial(\widehat{\mathbb{Q}\pi S|_E})$)

$\boxed{\Sigma_{0,n+1}}$ "symplectic expansion" compatible with (the unique) s

= "special expansion"



$1 \le k \le n$



$\pi = \pi_1(\Sigma_{0,n+1}, *)$.
$= \langle \gamma_1, \gamma_2, \cdots, \gamma_n \rangle$ free of rank n

$x_k := [\gamma_k] \in H = H_1(\Sigma_{0,n+1} : \mathbb{Q})$ $\quad 1 \le k \le n$

$x_0 := -\sum_{k=1}^{n} x_k = [\partial_0 S] \in H$

Definition

$\theta : \pi = \pi_1(\Sigma_{0,n+1}, *) \to \hat{T} = \hat{T}(H_1(\Sigma_{0,n+1} : \mathbb{Q}))$ special expansion

$\overset{\text{def}}{\Longleftrightarrow}$ 1) $\theta : \pi \to \hat{T}$ group-like expansion

2) (tangential) $1 \le \forall k \le n$, $\exists g_k \in \hat{T}$, s.t. $\Delta g_k = g_k \hat{\otimes} g_k$, $\theta(\gamma_k) = g_k e^{x_k} g_k^{-1}$

3) (special) $\theta(\gamma_1 \gamma_2 \cdots \gamma_n) = e^{-x_0} = e^{\sum_{k=1}^{n} x_k}$

examples 1) (Habegger - Masbaum) Kontsevich integral

2) $\left\{ \begin{array}{l} \text{analytic construction } /\mathbb{R} \quad (\text{K.}) \\ \text{combinatorial construction } /\mathbb{Q} \quad (\text{Kuno}) \end{array} \right.$

Action of $N(\hat{T})$ on $\hat{T}$

$u = \sum_{k=1}^{n} x_k u_k \in N(\hat{T}) \subset H \otimes \hat{T}$

$\sigma(u) \in \mathrm{Der}(\hat{T})$, $\quad \sigma(u)(x_k) := [u_k, x_k] \in \hat{T}$, $\quad 1 \le k \le n$,

<u>bracket on $N(\hat{T})$</u>

$$u,\ v = \sum_{k=1}^{m} x_k v_k \in N(\hat{T}) < H \otimes \hat{T}$$

$$[u,v] := -N\Big(\sum_{k=1}^{m} x_k [u_k, v_k]\Big) = \sum_{k=1}^{m} x_k \big(\sigma(u)(v_k) - \sigma(v)(u_k) - [u_k, v_k]\big) \quad (\because u, v \in N(\hat{T}))$$

<u>Theorem</u> (Kuno-K., Massuyeau-Turaev)

$S = \Sigma_{0,n+1}$, θ: special expansion

$\Rightarrow$ (1) $-N\theta : \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\cong} N(\hat{T})$ (top) isom. of Lie algebras

(2)

$$\begin{tikzcd}
\widehat{\mathbb{Q}\hat{\pi}} \times \widehat{\mathbb{Q}\pi} \arrow[r, "\sigma"] \arrow[d, "(-N\theta)\times\theta"', "\wr"] \arrow[dr, phantom, "\circlearrowright"] & \widehat{\mathbb{Q}\pi} \arrow[d, "\wr"] \\
N(\hat{T}) \times \hat{T} \arrow[r, "\sigma"'] & \hat{T}
\end{tikzcd}$$

$$\mathrm{sder}_n := \{ D : \hat{\mathcal{L}} \to \hat{\mathcal{L}} : \text{continuous derivation},\ {}^{1 \le \forall k \le n}_{\exists u_k \in \hat{\mathcal{L}}}\ D(x_k) = [u_k, x_k],\ D(x_0)\Big(= -\sum_{k=1}^{n} D(x_k)\Big) = 0 \}$$

special derivation Lie algebra

$N(H \otimes \hat{\mathcal{L}}) < N(\hat{T})$ Lie subalgebra annihilator of the coproduct Δ

$$0 \to \bigoplus_{k=1}^{m} \mathbb{Q} x_k^2 \to N(H \otimes \hat{\mathcal{L}}) \to \mathrm{sder}_n \to 0$$ central extension of Lie algebras (split)

$-x_k^2 \xleftarrow{-N\theta} \tau$ ((right-handed) Dehn twist along $\partial_k S$)

$$N(H \otimes \hat{\mathcal{L}}) \underset{-N\theta}{\cong} \{ u \in \mathbb{Q}\hat{\pi}(\Sigma_{0,n+1}) ;\ \Delta \circ \sigma(u) = (\sigma(u) \hat{\otimes} 1 + 1 \hat{\otimes} \sigma(u)) \circ \Delta \} =: (\widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{0,n+1}))_\Delta$$

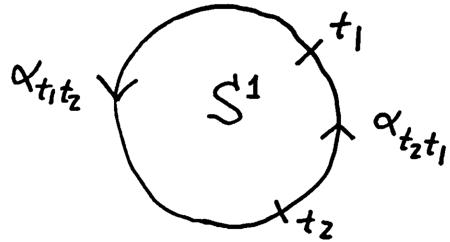
$$0 \to \bigoplus_{k=1}^{m} \mathbb{Q}\tau(\text{Dehn twist along } \partial_k S) \to (\widehat{\mathbb{Q}\hat{\pi}}(\Sigma_{0,n+1}))_\Delta \to \mathrm{sder}_n \to 0$$ (split) central extension

II. Turaev cobracket

$1 \in \hat{\pi} = \hat{\pi}(S)$ constant loop, $\mathbb{Z}\hat{\pi}' = \mathbb{Z}\hat{\pi}/\mathbb{Z}1$ quotient Lie algebra

$| \ |' : \mathbb{Z}\pi_1(S,p) \xrightarrow{| \ |} \mathbb{Z}\hat{\pi} \xrightarrow{\text{quotient}} \mathbb{Z}\hat{\pi}/\mathbb{Z}1 = \mathbb{Z}\hat{\pi}'$, $p \in S$

<u>Turaev cobracket</u>

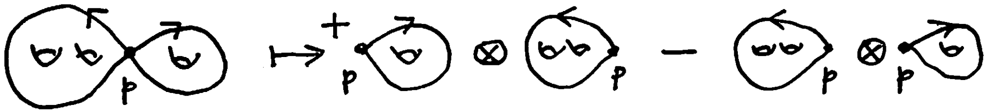


$\alpha \in \hat{\pi}$ in general position

$D_\alpha := \{(t_1, t_2) \in S^1 \times S^1 ;\ t_1 \neq t_2,\ \alpha(t_1) = \alpha(t_2)\}$

$$\delta(\alpha) \overset{\text{def}}{=} \sum_{(t_1,t_2) \in D_\alpha} \varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2))\, |\alpha_{t_1t_2}|' \otimes |\alpha_{t_2t_1}|' \in \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$$

<u>where</u> $\varepsilon(\dot{\alpha}(t_1), \dot{\alpha}(t_2)) \in \{\pm 1\}$ local intersection number



<u>Turaev</u>

(1) δ : well-defined

(2) $(\mathbb{Z}\hat{\pi}', [\ ,\ ], \delta)$: Lie bialgebra in the sense of Drinfel'd.

<u>Chas</u> : involutive

$\exists ! \ \delta : \widehat{\mathbb{Q}\hat{\pi}} \longrightarrow \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$ continuous extension

<u>Theorem</u> (Kuno-K.)

$\delta \circ \tau = 0 : \mathcal{M}(S)^0 \xrightarrow{\tau} \widehat{\mathbb{Q}\hat{\pi}} \xrightarrow{\delta} \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$

($\Leftarrow$ $\forall \varphi \in \mathcal{M}(S)$ preserves the self-intersections of any curves on S)

$\forall \theta : \pi \to \hat{T}$ group-like expansion

$$\begin{tikzcd}
\widehat{\mathbb{Q}\hat{\pi}} \arrow[r, "\delta"] \arrow[d, "-N\theta"', "\cong"] \arrow[dr, phantom, "\circlearrowright"] & \widehat{\mathbb{Q}\hat{\pi}} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}} \arrow[d, "\cong"', "(-N\theta)^{\hat{\otimes} 2}"] \\
N(\hat{T}) \arrow[r, "\exists ! \, \delta^\theta"'] & N(\hat{T}) \hat{\otimes} N(\hat{T})
\end{tikzcd}$$

δ^θ : tensorial description of δ

$$\delta^\theta = \sum_{k=-\infty}^{+\infty} \delta^\theta_{(k)} ,$$

$$\delta^\theta_{(k)} (N(H^{\otimes m})) \subset N(H^{\otimes (m+k)}) \quad (\forall m \geq 1)$$

$\boxed{\Sigma_{g,1}}$

<u>Theorem</u> (Massuyeau-Turaev, Kuno-K.)

$S = \Sigma_{g,1}$, θ: symplectic expansion

$\Rightarrow \delta^\theta = \delta^\theta_{(-2)} + \delta^\theta_{(0)} + \delta^\theta_{(1)} + \cdots\cdots$

$\delta^\theta_{(-2)}$ = Schedler's cobracket i.e. $\forall X_i \in H$

$$\delta^\theta_{(-2)} N(X_1 X_2 \cdots X_m) = - \sum_{j-i \geq 2} (X_i \cdot X_j) \{ N(X_{i+1} \cdots X_{j-1}) \hat{\otimes} N(X_{j+1} \cdots X_m X_1 \cdots X_{i-1}) - N(X_{j+1} \cdots X_m X_1 \cdots X_{i-1}) \hat{\otimes} N(X_{i+1} \cdots X_{j-1}) \}$$

($\Leftarrow$ Massuyeau-Turaev's tensorial description of the homotopy intersection form)

In their tensorial description, Massuyeau-Turaev introduced the "contraction operation"

$$\overset{\bullet}{\leadsto} : \hat{T}_{\geq 1} \times \hat{T}_{\geq 1} \to \hat{T}$$

defined by $(X_i, Y_j \in H,\ n, m \geq 1)$

$$X_1 \cdots X_{n-1} X_n \overset{\bullet}{\leadsto} Y_1 Y_2 \cdots Y_m := (X_n \cdot Y_1) X_1 \cdots X_{n-1} Y_2 \cdots Y_m$$

- $-\omega = -\sum_{i=1}^{g} (A_i B_i - B_i A_i)$: unit for the operation $\overset{\bullet}{\leadsto}$
- $\overset{\bullet}{\leadsto} |_{\hat{T}_{\geq 2} \times \hat{T}_{\geq 2}}$: associative

Corollary (Kuno-K.)

The Morita traces (= the Morita obstructions of the surjectivity of the Johnson homomorphisms) are included in the Turaev cobracket (except Tr^1)

$$\hat{\mathcal{L}} = \prod_{m=1}^{\infty} \mathcal{L}_m, \quad \mathcal{L}_m := \hat{\mathcal{L}} \cap H^{\otimes m}, \quad \hat{\mathcal{L}}_{\geq m} := \hat{\mathcal{L}} \cap \hat{T}_{\geq m}, \quad m \geq 1.$$

Morita traces, $m \geq 1$

$$\mathrm{Tr}^m : N(H \otimes \mathcal{L}_{m+1}) \subset H \otimes H \otimes H^{\otimes m} \xrightarrow{\cdot \otimes 1_{H^{\otimes m}}} H^{\otimes m} \to \mathrm{Sym}^m H$$

$$X_0 \otimes X_1 X_2 \cdots X_{m+1} \longmapsto (X_0 \cdot X_1) X_2 \cdots X_{m+1}$$

Morita

(1) m: even $\Rightarrow \mathrm{Tr}^m(N(H \otimes \mathcal{L}_{m+1})) = 0$

(2) m: odd $\Rightarrow \mathrm{Tr}^m(N(H \otimes \mathcal{L}_{m+1})) = \mathrm{Sym}^m H$

(3) $m = 1$ $\mathrm{Tr}^1 \circ$ (the 1st Johnson homomorphism) $\neq 0$

(4) $m \geq 3$, odd $\mathrm{Tr}^m \circ$ (the mth Johnson homomorphism) $= 0$

refinement

Enomoto-Satoh trace

$$\widehat{\mathrm{Tr}} : N(H\otimes\hat{\mathcal{L}}_{\geq 2}) \subset H\otimes H\otimes \hat{T} \longrightarrow \hat{T} \xrightarrow{N} N(\hat{T}) \overset{S}{\cong} \hat{T}/\overline{(\mathbb{Q}1+[\hat{T},\hat{T}])}$$

$$X_0X_1X_2\cdots X_{m+1} \longmapsto (X_0\cdot X_1)X_2\cdots X_{m+1} \longmapsto (X_0\cdot X_1)N(X_2\cdots X_{m+1})$$

$$(X_0\cdot X_1)X_2\cdots X_{m+1} \in \hat{T}/\overline{(\mathbb{Q}1+[\hat{T},\hat{T}])}$$

Enomoto

The Enomoto-Satoh trace are <u>not</u> included in Schedler's cobracket $\delta^{\theta}_{(-2)}$

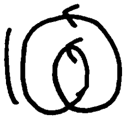
$\Downarrow$

Regular homotopy version of the Goldman-Turaev Lie bialgebra

$\hat{\pi}^+ = \hat{\pi}^+(S) = \{ C^\infty \text{ immersion } S^1 \to S \}/\text{regular homotopy}$

f : framing of TS

$\mathrm{rot}_f : \hat{\pi}^+ \to \mathbb{Z}$ rotation number w.r.to f

$\mathrm{rot}_f($  $) = 2$ $\quad$ $\mathrm{rot}_f($  $) = 0$

$\hat{\pi}^+ \cong \hat{\pi} \times \langle r\rangle$, $\alpha \longmapsto (\alpha, r^{\mathrm{rot}_f\alpha})$, $\quad r$: formal parameter, $\langle r\rangle \cong \mathbb{Z}$

$\mathbb{Z}\hat{\pi}^+ \cong \mathbb{Z}\hat{\pi} \otimes_{\mathbb{Z}} \mathbb{Z}\langle r\rangle$, $\quad \mathbb{Z}\langle r\rangle = \mathbb{Z}[r, r^{-1}]$ Laurent polynomials

$\mathbb{Z}\langle r\rangle$-module structure on $\mathbb{Z}\hat{\pi}^+$ is independent of the choice of f

$[\ ,\ ]^+ : \mathbb{Z}\hat{\pi}^+ \underset{\mathbb{Z}\langle r\rangle}{\otimes} \mathbb{Z}\hat{\pi}^+ \to \mathbb{Z}\hat{\pi}^+$, regular homotopy version of the Goldman bracket

$\delta^+ : \mathbb{Z}\hat{\pi}^+ \to \mathbb{Z}\hat{\pi}^+ \underset{\mathbb{Z}\langle r\rangle}{\otimes} \mathbb{Z}\hat{\pi}^+$, regular homotopy version of the Turaev cobracket

$\theta : \pi \to \widehat{T}$: group-like expansion

f : framing of TS

$$\Rightarrow N^+\theta_f : \begin{pmatrix} \text{completion} \\ \text{of } \mathbb{Q}\hat{\pi}^+ \end{pmatrix} \xrightarrow{\cong} N^+(\widehat{T}) \hat{\otimes} \mathbb{Q}[[\rho]]$$

<u>where</u> $\rho = \log r$, $\widehat{\mathbb{Q}\langle r\rangle} = \mathbb{Q}[[\rho]]$

$N^+ : \widehat{T} \to \widehat{T}$ continuous linear map

$N^+|_{H^{\otimes 0}} := 1_{H^{\otimes 0}} (\neq N|_{H^{\otimes 0}})$

$N^+|_{H^{\otimes m}} := N|_{H^{\otimes m}}$ for $m \geq 1$

$$N^+(\widehat{T}) = \mathbb{Q}1 \oplus N(\widehat{T})$$

$$\delta^{+,\theta_f} : N^+(\widehat{T}) \hat{\otimes} \mathbb{Q}[[\rho]] \to N^+(\widehat{T}) \hat{\otimes} N^+(\widehat{T}) \hat{\otimes} \mathbb{Q}[[\rho]]$$

tensorial description of δ^+

Theorem (K.) $S=\Sigma_{g,1}$

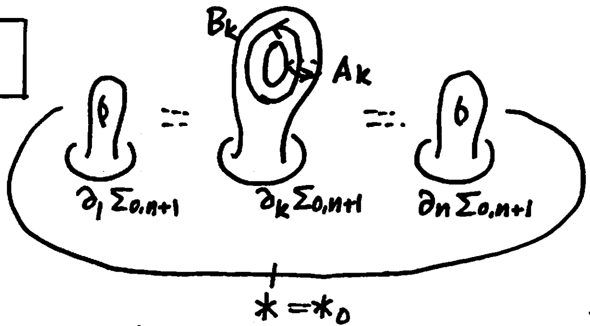
θ : symplectic expansion , f : framing of $T\Sigma_{g,1}$

$\Rightarrow$ The lowest degree term of δ^{+,θ_f} is given by $(X_j \in H)$

$$\delta^{+,\theta_f}_{(-2)}(N^+(X_1\cdots X_m))$$
$$=\sum_{j-i\geq 1}(X_i\cdot X_j)\left\{N^+(X_{i+1}\cdots X_{j-1})\hat{\otimes}N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1})-N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1})\hat{\otimes}N^+(X_{i+1}\cdots X_{j-1})\right\}$$

In particular, the $N^+(\hat{T})\hat{\otimes}N^+(1)$-component of $\delta^{+,\theta_f}_{(-2)}$ equals the Enomoto-Satoh trace.

$\boxed{\Sigma_{0,n+1}}$



capping $\Sigma_{1,1}$ on each $\partial_k\Sigma_{0,n+1}$, $1\leq k\leq n$, to obtain $\Sigma_{0,n+1}\hookrightarrow\Sigma_{n,1}$

$$\iota:\hat{T}=\hat{T}(H_1(\Sigma_{0,n+1};\mathbb{Q}))\to\hat{T}(H_1(\Sigma_{n,1};\mathbb{Q}))$$
$$x_k\longmapsto A_kB_k-B_kA_k \quad (1\leq k\leq n)$$

injective homomorphism of complete Hopf algebras

pull-back of Massuyeau-Turaev's contraction operation $\dot{\leadsto}$ by ι

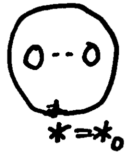
$\dot{\leadsto}:\hat{T}_{\geq 1}\times\hat{T}_{\geq 1}\to\hat{T}_{\geq 1}$

$$x_{i_1}\cdots x_{i_{\ell-1}}x_{i_\ell}\dot{\leadsto}x_{j_1}x_{j_2}\cdots x_{j_m}=-\delta_{i_\ell j_1}x_{i_1}\cdots x_{i_{\ell-1}}x_{j_2}\cdots x_{j_m}\quad(1\leq i_1,\cdots,i_\ell,j_1,\cdots,j_m\leq n,\ \ell,m\geq 1)$$

$x_0=-\sum_{k=1}^{n}x_k$: unit for $\dot{\leadsto}$

$\dot{\leadsto}$: associative

Theorem (K.) $(S=\Sigma_{0,n+1})$

θ: special expansion, f: framing of $T\Sigma_{0,n+1}$ coming from the embedding 

$\hookrightarrow \mathbb{R}^2$

$\Longrightarrow$ The value at $p=0$ of the lowest degree term of δ^{+,θ_f} is given by

$$\delta^{+,\theta_f}_{(-1)}(N^+(X_1\cdots X_m))\big|_{p=0}$$
$$=\sum_{j-i\geq 1}\Big\{N^+(X_i\rightsquigarrow X_j\,X_{j+1}\cdots X_m X_1\cdots X_{i-1})\,\hat{\otimes}\,N^+(X_{i+1}\cdots X_{j-1})$$
$$+N^+(X_j\rightsquigarrow X_i\,X_{i+1}\cdots X_{j-1})\,\hat{\otimes}\,N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1})$$
$$-N^+(X_{i+1}\cdots X_{j-1})\,\hat{\otimes}\,N^+(X_i\rightsquigarrow X_j\,X_{j+1}\cdots X_m X_1\cdots X_{i-1})$$
$$-N^+(X_{j+1}\cdots X_m X_1\cdots X_{i-1})\,\hat{\otimes}\,N^+(X_j\rightsquigarrow X_i\,X_{i+1}\cdots X_{j-1})\Big\}$$

In particular, the $N^+(1)\hat{\otimes}N^+(\hat{T})$-component

= the divergence cocycle in the Kashiwara-Vergne problem

- In the formulation by Alekseev-Torossian, any solution to the Kashiwara-Vergne problem can be regarded as a special expansion for $\Sigma_{0,3}$ with an extra condition involved with the divergence cocycle and the Bernoulli numbers

Expectation (#)

$\forall\theta$: solution to the Kashiwara-Vergne problem

$\delta^\theta \overset{?}{=} \delta^\theta_{(-1)}$, the lowest degree term of δ^θ

<u>More generally</u> $S = \Sigma_{g,n+1}$, $(g, n \geq 0)$, $E \subset \partial S$

(<u>Question</u> (##) Does there exist a "symplectic expansion" of $\pi S|_E$ s.t
$\bar{\delta}^\theta$ = the lowest degree term of δ^θ ?

<u>Kuno</u>
(1) $\exists\, \theta$: symplectic expansion for $\Sigma_{g,1}$ s.t. $\delta^\theta \neq \delta^\theta_{(-2)}$
(2) $(\Sigma_{1,1})$ $\exists\, \theta$: symplectic expansion for $\Sigma_{1,1}$. s.t. $\delta^\theta \equiv \delta^\theta_{(-2)}$ modulo degree ≥ 9.

If the expectation (#) is affirmative, then the question (##) may be regarded as a positive genus analogue of the Kashiwara-Vergne problem
If (#) and (##) are negative, we obtain candidates for new obstructions for the surjectivity of the Johnson homomorphisms.

<u>Bernoulli numbers</u>

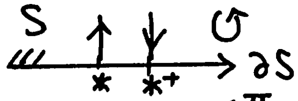
$$g(z) := \frac{z}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^m = 1 - \frac{1}{2} z + \sum_{g=1}^{\infty} \frac{B_{2g}}{(2g)!} z^{2g}$$

, B_m: Bernoulli number $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$ ...

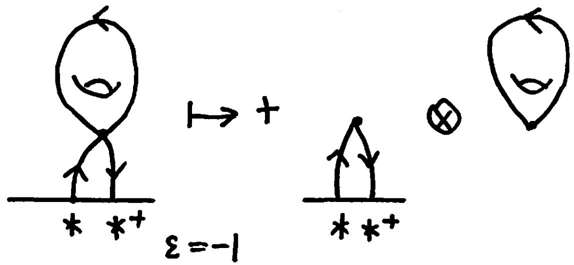
The Bernoulli numbers appear in
① (Massuyeau-Turaev) tensorial description of the homotopy intersection form
② (Fukuhara-Kuno-K.) self-intersection of log (simple closed curve)

Coaction of $\mathbb{Z}\hat{\pi}'(S)$ on $\mathbb{Z}\pi_1(S)$ (inspired by Turaev's μ)

$*, *^+ \in \partial S$



$\gamma \in \Pi S(*, *^+) \cong \pi_1(S, *) = \pi$ in general position



$\Gamma_\gamma := \{$ self-intersection point of $\gamma \}$

$\ni p, \quad 0 < {}^{\exists 1} t_1^p < {}^{\exists 1} t_2^p < 1 \qquad \gamma(t_1^p) = \gamma(t_2^p) = p$

$$\mu(\gamma) \overset{\mathrm{def}}{=} -\sum_{p \in \Gamma_\gamma} \varepsilon(\dot{\gamma}(t_1^p), \dot{\gamma}(t_2^p)) \, (\gamma_{0 t_1^p} \gamma_{t_2^p 1}) \otimes |\gamma_{t_1^p t_2^p}|' \in \mathbb{Z}\pi \otimes \mathbb{Z}\hat{\pi}'$$

$$\delta \circ |\ |' = \mathrm{alt} \circ (|\ |' \otimes 1) \circ \mu : \mathbb{Z}\pi \to \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}'$$

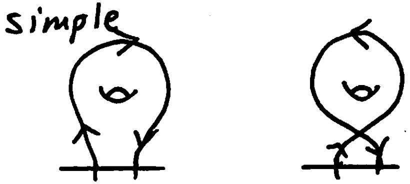
$$\mathrm{alt} : \mathbb{Z}\hat{\pi}' \otimes \mathbb{Z}\hat{\pi}' \circlearrowleft \qquad u \otimes v \mapsto u \otimes v - v \otimes u$$

Kuno-K.

(1) μ : well-defined

(2) $(\mathbb{Z}\pi, \sigma, \mu)$ is a $(\mathbb{Z}\hat{\pi}', [\ ,\ ], \delta)$-bimodule

$\exists !\ \mu : \widehat{\mathbb{Q}\pi} \to \widehat{\mathbb{Q}\pi} \hat{\otimes} \widehat{\mathbb{Q}\hat{\pi}}$ continuous extension



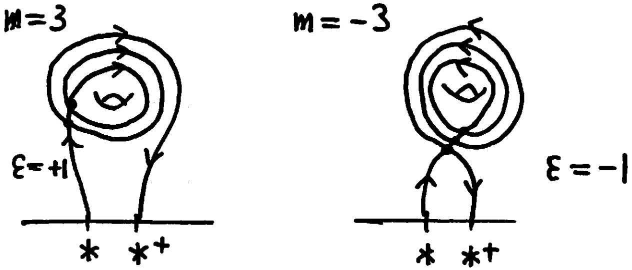
<u>Theorem</u> (Fukuhara - Kuno - K.)

$\gamma \in \Pi S(*, *^+)$ simple path

$$\mu(\log\gamma) = \frac{1}{2}\, 1 \hat{\otimes} |\log\gamma|' - \sum_{g=1}^{\infty} \frac{B_{2g}}{(2g)!} \sum_{p=0}^{2g-1} \binom{2g}{p} (-1)^p (\log\gamma)^p \hat{\otimes} |(\log\gamma)^{2g-p}|'$$

<u>proof</u> $m \in \mathbb{Z}$

$$\mu(\gamma^m) = \begin{cases} -\sum_{k=1}^{m(-1)} \gamma^k \otimes |\gamma^{m-k}|', & \text{if } m>0 \\ 0, & \text{if } m=0 \\ +\sum_{k=0}^{|m|-1} \gamma^{-k} \otimes |\gamma^{m+k}|', & \text{if } m<0 \end{cases}$$



$k=m$
$|\gamma^{m-k}|' = |1|' = 0$

$\mathbb{Q}[[Z]]$ formal power series

$z := e^Z \in \mathbb{Q}[[Z]]$, $\mathbb{Q}[z, z^{-1}] \subset \mathbb{Q}[[Z]]$. Laurent polynomials

$\hat{\mu} : \mathbb{Q}[z, z^{-1}] \to \mathbb{Q}[z, z^{-1}] \otimes \mathbb{Q}[z, z^{-1}]$

$$\hat{\mu}(z^m) \overset{\text{def}}{=} \begin{cases} -\sum_{k=1}^{m} z^k \otimes z^{m-k}, & \text{if } m>0 \\ 0, & \text{if } m=0 \\ +\sum_{k=0}^{|m|-1} z^{-k} \otimes z^{m+k}, & \text{if } m<0 \end{cases}$$

$$\Rightarrow (z^{-1} \otimes z - 1)\, \hat{\mu}(z^m) = -1 \otimes z^m + z^m \otimes 1$$

$\forall f(z) \in \mathbb{Q}[z, z^{-1}]$

$$(z^{-1} \otimes z - 1)\,\hat{\mu}(f(z)) = -1 \otimes f(z) + f(z) \otimes 1$$

$\Longrightarrow \quad \exists!\ \hat{\mu} : \mathbb{Q}[[Z]] \longrightarrow \mathbb{Q}[[Z]] \hat{\otimes} \mathbb{Q}[[Z]]$ continuous extension

In particular, if $f(z) = \log z = Z$, then

$$\hat{\mu}(\log z) = \hat{\mu}(Z) = \frac{-1 \hat{\otimes} Z + Z \hat{\otimes} 1}{z^{-1} \hat{\otimes} z - 1} = \frac{-(-Z \hat{\otimes} 1 + 1 \hat{\otimes} Z)}{e^{-Z \hat{\otimes} 1 + 1 \hat{\otimes} Z} - 1} = -g(-Z \hat{\otimes} 1 + 1 \hat{\otimes} Z)$$

$$= -\sum_{m=0}^{\infty} \frac{B_m}{m!} \sum_{p=0}^{m} \binom{m}{p} (-1)^p Z^p \hat{\otimes} Z^{m-p} \qquad // \text{Thm}$$

$\Longrightarrow \quad \theta^{std} : \pi = \pi_1(\Sigma_{0,n+1}, *) \longrightarrow \hat{T} = \hat{T}(H_1(\Sigma_{0,n+1}; \mathbb{Q}))$

$\gamma_k \longmapsto e^{x_k}$ standard group-like expansion (<u>not</u> special)

<u>K.</u> Complete tensorial description of $\mu^{\theta^{std}}$ and $\delta^{\theta^{std}}$ (--- too complicated !!)

Another application of the map $\hat{\mu}$

<u>Theorem</u> (Fukuhara - Kuno - K.) $0 \le a \le m \le n$, $m \ge 2$, integers

$$\Longrightarrow \quad B_m = (-1)^a \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} \sum_{i=1}^{k-1} i^a (k-i)^{m-a}$$

In particular, if $a = m = n$,

<u>Corollary</u> (Kronecker) $\forall m \ge 2$, $\quad B_m = \sum_{k=1}^{m+1} \frac{(-1)^{k+1}}{k} \binom{m+1}{k} \sum_{i=1}^{k-1} i^m$ —