Rational Homotopy and Intrinsic Formality of E_n-operads Part II

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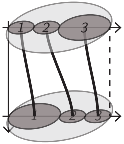

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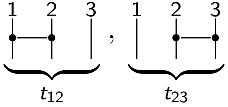

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Plan

- Part I:
  - §0. The notion of an operad
  - §1. The little n-discs operads
  - §2. The (co)homology of the little discs operads
  - §3. The rational homotopy theory of operads
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- Part II:
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§1. The Drinfeld-Kohno Lie algebra operad

- Definition: The rth Drinfeld-Kohno Lie algebra $\hat{\mathfrak{p}}(r)$ is defined by the presentation:

$$\hat{\mathfrak{p}}(r) = \hat{\mathbb{L}}(t_{ij}, 1 \leq i \neq j \leq r)/\langle [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}] \rangle$$

  where:
  - a generator t_{ij} such that $t_{ij} = t_{ji}$ is assigned to each pair $1 \leq i \neq j \leq r$,
  - the commutation relations

    $$[t_{ij}, t_{kl}] \equiv 0$$

    hold for all quadruple of pair-wise distinct indices $i \neq j \neq k \neq l$,
  - and the Yang-Baxter relations

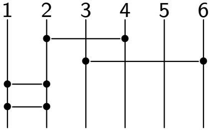
    $$[t_{ij}, t_{ik} + t_{jk}] = 0.$$

    hold for all triple of pair-wise distinct indices $i \neq j \neq k \neq l$.
- Remark: We have

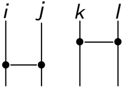
$$\hat{\mathbb{U}}(\hat{\mathfrak{p}}(r)) = \hat{\mathbb{T}}(t_{ij}, 1 \leq i \neq j \leq r)/\langle [t_{ij}, t_{kl}], [t_{ij}, t_{ik} + t_{jk}] \rangle$$

  with $[u, v] = uv - \pm vu$.

The monomials $t_{i_1 j_1} \cdots t_{i_r j_r} \in \hat{\mathbb{U}}(\hat{\mathfrak{p}}(r))$ are usually represented by chord diagrams on r strands. For instance:

$t_{12} t_{12} t_{36} t_{24} =$  .

In this representation, the commutator and Yang-Baxter relations read:



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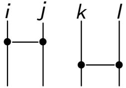
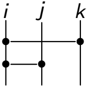


$= 0,$



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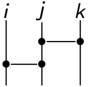

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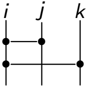

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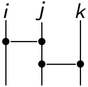
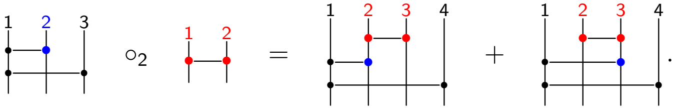


$= 0.$

- Observation: The Drinfeld-Kohno Lie algebras inherit the structure of an operad in the category of complete Lie algebras $\hat{\mathfrak{p}}$, and the associated algebras of chord diagrams form an operad in the category of Hopf algebras $\hat{\mathbb{U}}\hat{\mathfrak{p}}$:
  - the symmetric group Σ_r acts on $\hat{\mathfrak{p}}(r)$ by permutation of strand indices;
  - the composition operations

$$\hat{\mathfrak{p}}(k) \oplus \hat{\mathfrak{p}}(l) \xrightarrow{\circ_i} \hat{\mathfrak{p}}(k+l-1)$$
$$\Leftrightarrow \hat{\mathbb{U}}(\hat{\mathfrak{p}}(k)) \otimes \hat{\mathbb{U}}(\hat{\mathfrak{p}}(l)) \xrightarrow{\circ_i} \hat{\mathbb{U}}(\hat{\mathfrak{p}}(k+l-1)),$$

    are given by the following insertion operations (in the chord diagram picture):



§2. The realization of the $(n-1)$-Poisson cooperad

- Reminder: We have

$$\mathrm{H}^*(\mathrm{D}_2(r)) = \mathrm{H}^*(\mathrm{F}(\mathring{\mathbb{D}}^2, r)) = \frac{\mathbb{S}(\omega_{ij}, 1 \leq i \neq j \leq r)}{(\omega_{ij}^2, \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij})}$$

  where $\deg(\omega_{ij}) = 1$ and $\omega_{ij} = \omega_{ji}$ for each pair $i \neq j$.

- Theorem (Kohno): Let:

  $\mathrm{C}^*_{CE}(\hat{\mathfrak{p}}(r)) =$ Chevalley-Eilenberg cochain complex $= (\mathbb{S}(\Sigma^{-1}\hat{\mathfrak{p}}(r)^\vee), \partial)$,

  We have a quasi-isomorphism of commutative dg-algebras

$$\kappa : \mathrm{C}^*_{CE}(\hat{\mathfrak{p}}(r)) \xrightarrow{\sim} \mathrm{H}^*(\mathrm{F}(\mathring{\mathbb{D}}^2, r))$$

  given by the following mapping:

$$\begin{cases} \kappa(t_{ij}^\vee) = \omega_{ij}, & \text{for each pair } i \neq j, \\ \kappa(\pi^\vee) = 0, & \text{when } \pi \text{ has weight } m > 1. \end{cases}$$

  for each arity $r \in \mathbb{N}$.

- Observation (Tamarkin): The dg-algebras $\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}(r))$ inherit composition coproducts

$$\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}(k+l-1)) \xrightarrow{\circ_i^*} \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}(k) \oplus \hat{\mathfrak{p}}(l)) \xleftarrow{\simeq} \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}(k)) \otimes \mathtt{C}^*_{CE}(\hat{\mathfrak{p}}(l))$$

and form a Hopf dg-cooperad. Furthermore, the Kohno map defines a quasi-isomorphism of Hopf dg-cooperads

$$\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}) \xrightarrow{\sim} \mathtt{H}^*(\mathsf{D}_2).$$

- Observation: This result extends to all operads D_n, for a graded version of the Drinfeld-Kohno Lie algebra operad $\hat{\mathfrak{p}}_n$ defined by the same presentation, with $\deg(t_{ij}) = n-2$ and $t_{ij} = (-1)^n t_{ij}$ as unique changes.

- Proposition: The Hopf cooperad $\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n)$ defines a cofibrant resolution of the cohomology cooperad $\mathtt{H}^*(\mathsf{D}_n) = \mathsf{Pois}^c_{n-1}$.
- Corollary: We have

$$\begin{aligned} & \langle \mathsf{Pois}^c_{n-1}(r)\rangle = \mathtt{Mor}_{dg\,\mathcal{C}om}(\mathtt{C}^*_{CE}(\hat{\mathfrak{p}}_n(r)), \Omega^*(\Delta^\bullet))) \\ \Rightarrow \quad & \langle \mathsf{Pois}^c_{n-1}(r)\rangle = \mathtt{MC}_\bullet(\hat{\mathfrak{p}}_n(r)) \end{aligned}$$

where $\mathtt{MC}_\bullet(\hat{\mathfrak{p}}_n(r))$ is the simplicial set of forms

$$\gamma \in \hat{\mathfrak{p}}_n(r) \hat{\otimes} \Omega^*(\Delta^\bullet)$$

such that $\deg^*(\gamma) = 1$ and:

$$\delta(\gamma) + \frac{1}{2}[\gamma, \gamma] = 0.$$

- Remark: In the case of the standard Drinfeld-Kohno Lie algebra operad $\hat{\mathfrak{p}} = \hat{\mathfrak{p}}_2$, we have:

$$\mathtt{MC}_\bullet(\hat{\mathfrak{p}}(r)) \sim \mathtt{B}(\mathbb{G}\,\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)) = \mathtt{B}(\exp\hat{\mathfrak{p}}(r)),$$

where $\mathtt{B}$:= classifying space functor from groups to spaces.

§3. The obstruction theory proof of the intrinsic formality theorem

- Idea: Use the model $\mathtt{G}_\bullet : dg\,\mathcal{H}opf\,\mathcal{O}p^c \rightleftarrows \mathcal{S}imp\mathcal{O}p^{op} : \Omega^*_\sharp$ to transport the problem in the category of Hopf dg-cooperads.
- Goal: Let K be a Hopf dg-cooperad such that we have:
  - a cohomology isomorphism $\mathtt{H}^*(\mathsf{K}) \simeq \mathsf{Pois}^c_{n-1}$, for some $n \geq 3$,
  - an involutive isomorphism $J : \mathsf{K} \xrightarrow{\simeq} \mathsf{K}$ which mirrors the action of a hyperplane reflection on D_n in the case $4 \mid n$.

  Pick:
  - a fibrant resolution of this Hopf dg-cooperad $\mathsf{K} \xrightarrow{\sim} \mathtt{Res}_{\mathcal{O}p}(\mathsf{K}) =: \mathsf{Q}$,
  - a cofibrant resolution of the Poisson cooperad $\mathsf{R} := \mathtt{Res}^{\mathcal{C}om}(\mathsf{Pois}^c_{n-1}) \xrightarrow{\sim} \mathsf{Pois}^c_{n-1}$,

  and prove the existence of a morphism of Hopf dg-cooperads:

$$\mathsf{Pois}^c_{n-1} \xleftarrow{\sim} \mathtt{Res}^{\mathcal{C}om}(\mathsf{Pois}^c_{n-1}) \underset{\exists ?}{\overset{\sim}{\dashrightarrow}} \mathtt{Res}_{\mathcal{O}p}(\mathsf{K}) \xrightarrow{\sim} \mathsf{K}$$

  which induces $\mathtt{H}^*(\mathsf{K}) \simeq \mathsf{Pois}^c_{n-1}$.

- Definition: Let

$$\mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{R},\mathsf{Q}) = \{\text{Hopf dg-cooperads maps } \phi : \mathsf{R} \to \mathsf{Q}^{\Delta^\bullet}\}$$

so that $\mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{R},\mathsf{Q})_0 = \mathtt{Mor}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{R},\mathsf{Q})$.

- Constructions: Take $\mathtt{Res}_{\mathcal{O}p}(\mathsf{K}) := \mathtt{Tot}(\mathtt{Res}^\bullet_{\mathcal{O}p}(\mathsf{K}))$, where

$$\mathsf{R}^\bullet = \mathtt{Res}^\bullet_{\mathcal{O}p}(\mathsf{K}) := \text{operadic triple coresolution of } \mathsf{K},$$

and $\mathtt{Res}^{\mathcal{C}om}(\mathsf{Pois}^c_{n-1}) := |\,\mathtt{Res}^{\mathcal{C}om}_\bullet(\mathsf{Pois}^c_{n-1})|$, where

$$\mathsf{Q}_\bullet = \mathtt{Res}^{\mathcal{O}p}_\bullet(\mathsf{H}) := \text{cotriple resolution of } \mathsf{H} = \mathsf{Pois}^c_{n-1} \text{ in } dg\,\mathcal{C}om.$$

- Observation: We have
$\mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(|\,\mathsf{R}_\bullet\,|, \mathtt{Tot}(\mathsf{Q}^\bullet)) = \mathtt{Tot}\,\mathtt{Diag}(X^{\bullet\bullet})$ where

$$X^{\bullet\bullet} = \mathtt{Map}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{R}_\bullet, \mathsf{Q}^\bullet).$$

- Theorem (Bousfield): The obstructions to the existence of
$\phi \in \mathtt{Tot}(X^{\bullet\bullet})_0 \Leftrightarrow \phi \in \mathtt{Mor}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{R},\mathsf{Q})$ lie in $\pi^{s+1}\pi_s(X^{\bullet\bullet})$.

- Theorem (BF, Willwacher): We have a sequence of reductions:

$$\begin{aligned}\pi^*\pi_*(X^{\bullet\bullet}) &\simeq \mathrm{H}^* \mathtt{BiDer}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathtt{Res}_{\bullet}^{\mathcal{C}om}(\mathsf{Pois}_{n-1}^c), \mathtt{Res}_{\mathcal{O}p}^{\bullet}(\mathsf{Pois}_{n-1}^c))\\ &\simeq \mathrm{H}^*(\mathtt{BiDef}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{Pois}_{n-1}^c, \mathsf{Pois}_{n-1}^c)\\ &\simeq \mathrm{H}^*(\mathtt{BiDef}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{Graph}_n^c, \mathsf{Pois}_{n-1}^c)\\ &\simeq \mathrm{H}^*(\mathbb{Q}\ltimes \mathsf{GC}_n^2)\end{aligned}$$

where:

  - $\mathtt{BiDef}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(-,-)$ is an analogue for Hopf dg-cooperads of the Gerstenhaber-Schack deformation bicomplex of bialgebras,
  - Graph_n^c is an operad of graphs such that $\mathsf{Graph}_n^c \xrightarrow{\sim} \mathsf{Pois}_{n-1}^c$.
  - GC_n^2 is Kontsevich's complex of graphs with at least bivalent vertices.

- Observation: For $n \geq 3$, we have $\mathrm{H}^1(\mathbb{Q}\ltimes \mathsf{GC}_n^2) = 0$ when $n \not\equiv 0(4)$, while we have $\mathrm{H}^1(\mathbb{Q}\ltimes \mathsf{GC}_n^2)^{J_*} = 0$ for all n, for an involution J_* inherited from $\mathsf{Pois}_{n-1}^c \simeq \mathrm{H}^*(\mathsf{D}_n) \simeq \mathrm{H}^*(\mathsf{K})$ and $\mathsf{Graph}_n^c \xrightarrow{\sim} \mathsf{Pois}_{n-1}^c$.

- Corollary: The obstructions to the existence of a map $\phi \in \mathtt{Mor}_{dg\,\mathcal{H}opf\,\mathcal{O}p^c}(\mathsf{R},\mathsf{Q})$ vanish.

References

- *The intrinsic formality of E_n-operads*, with Thomas Willwacher
  `Preprint arXiv:1503.08699`
- *Homotopy of Operads and Grothendieck-Teichmüller Groups*
  `http://math.univ-lille1.fr/~fresse/OperadHomotopyBook`

Homotopy of Operads and Grothendieck-Teichmüller Groups

- Part I. From operads to Grothendieck-Teichmüller groups
  1. Introduction to the general theory of operads
  2. Braids and E_2-operads
  3. Hopf algebras and the Malcev completion
  4. The operadic definition of (pro-unipotent) Grothendieck-Teichmüller groups
- Part II. The applications of homotopy theory to operads
  1. Introduction to general homotopy theory methods
  2. Modules, algebras and the rational homotopy of spaces
  3. The rational homotopy of operads
  4. Applications of the rational homotopy to E_n-operads
- Part III. The computation of homotopy automorphism spaces of operads
  1. The applications of homotopy spectral sequences
  2. The case of E_n-operads
- Appendices.
  1. The construction of free operads
  2. The cotriple resolution of operads
  3. Cofree cooperads and the bar and Koszul duality of operads

Thank you for your attention!

Appendix. The fundamental groupoid of the little 2-discs operad

- The homotopy equivalence $\mathsf{D}_2(r) \sim \mathsf{F}(\mathring{\mathbb{D}}^2, r)$ implies that the space $\mathsf{D}_2(r)$ is an Eilenberg-MacLane space $\mathtt{K}(P_r, 1)$, where $P_r = \pi_1 \mathsf{F}(\mathring{\mathbb{D}}^2, r)$ is the pure braid group on r strands.
- The idea is to use the fundamental groupoids $\pi\,\mathsf{D}_2(r)$ in order to get a combinatorial model of the operad D_2. We have

$$\mathsf{D}_2 \sim \mathtt{B}(\pi\,\mathsf{D}_2),$$

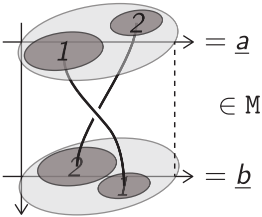
where $\mathtt{B}$:= classifying space functor from groupoids to spaces.

Construction:

- The fundamental groupoid $\pi\,\mathrm{D}_2(r)$ has $\mathtt{Ob}\,\pi\,\mathrm{D}_2(r) = \mathrm{D}_2(r)$ as object set.
- The morphisms $\mathtt{Mor}_{\pi\,\mathrm{D}_2(r)}(\underline{a}, \underline{b})$ are homotopy classes of paths $\gamma : [0,1] \to \mathrm{D}_2(r)$ going from $\gamma(0) = \underline{a}$ to $\gamma(1) = \underline{b}$.
- The map

$$\mathtt{Mor}_{\pi\,\mathrm{D}_2(r)}(\underline{a}, \underline{b}) \xrightarrow[\simeq]{\text{disc centers}} \mathtt{Mor}_{\pi\,\mathrm{F}(\mathring{\mathbb{D}}^2, r)}(\underline{a}, \underline{b})$$

identifies the morphism sets of this groupoid with cosets of the pure braid group P_r inside the braid group B_r. Thus, a morphism in this groupoid can be represented by a picture of the form:

$\gamma =$  $\in \mathtt{Mor}\,\pi\,\mathrm{D}_2(2)$

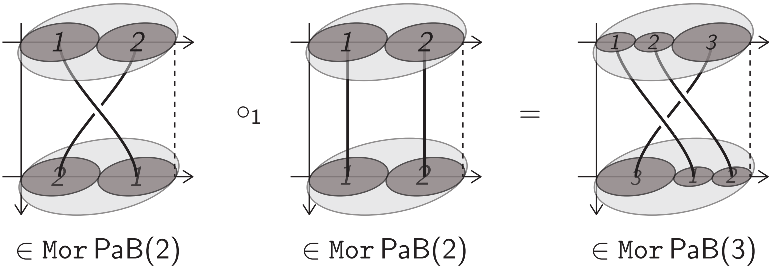
The structure of the fundamental groupoid operad:

- The groupoids $\pi\,\mathrm{D}_2(r)$ inherit a symmetric structure, as well as operadic composition products

$$\circ_i : \pi\,\mathrm{D}_2(k) \times \pi\,\mathrm{D}_2(l) \to \pi\,\mathrm{D}_2(k+l-1),$$

and hence form an operad in the category of groupoids.

- In the braid picture, the operadic composition products can be depicted as cabling operations:

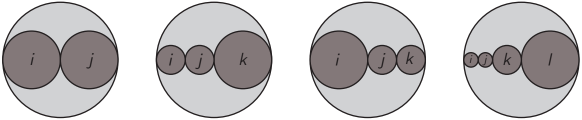


Ideas:

- There is no need to consider the whole $\mathsf{D}_2(r)$ as object set.
- The groupoids of parenthesized braids $\mathsf{PaB}(r)$ are full subgroupoids of the fundamental groupoid $\pi\,\mathsf{D}_2(r)$ defined by appropriate subsets of little 2-discs configurations $\Omega(r) \subset \mathsf{D}_2(r)$ as object sets.
- These object sets $\Omega(r)$ are preserved by the operadic composition structure of little 2-discs so that the collection of groupoids $\mathsf{PaB}(r)$, $r \in \mathbb{N}$, forms a suboperad of $\pi\,\mathsf{D}_2$.

Construction:

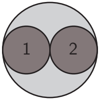
- The sets $\Omega(r)$, $r = 2, 3, 4, \dots$, prescribing the origin and end-points of paths in $\mathrm{PaB}(r)$, consist of little 2-disc configurations of the following form:



$\cdots$

(the indices $i, j, \dots$ run over all permutations of $1, 2, \dots$).

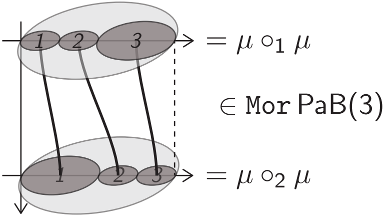
- These configurations represents the iterated operadic composites of the following element

$\mu =$  $\in \mathrm{D}_2(2)$

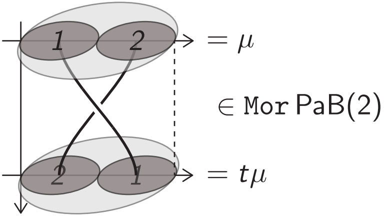
and the sets $\Omega(r)$, $r \in \mathbb{N}$, are the components of the suboperad of D_2 generated by this element. *This operad is free.*

The operad PaB is generated by the following fundamental morphisms:

- the associator

$\alpha =$ 

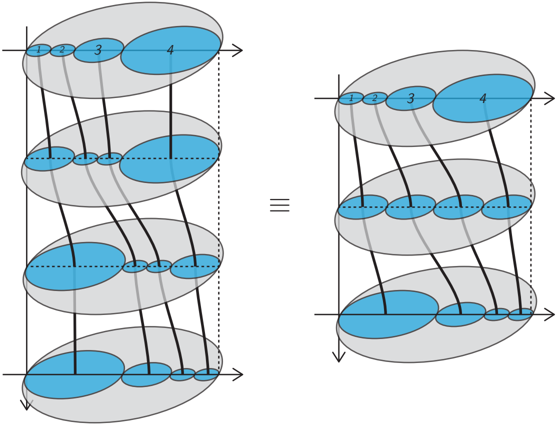
- and the braiding

$\tau =$ 

where $t = (12) \in \Sigma_2$.

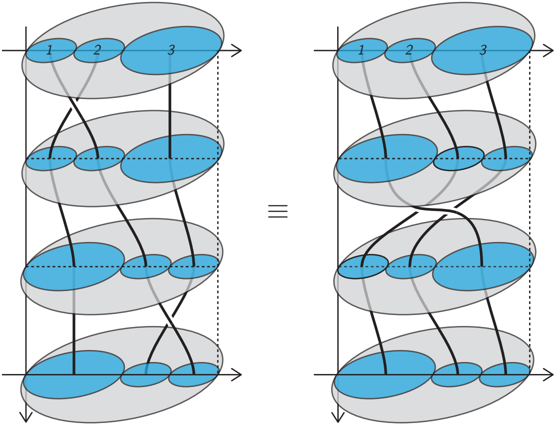
In the morphism set of the operad PaB:

- the associator satisfy the pentagon equation

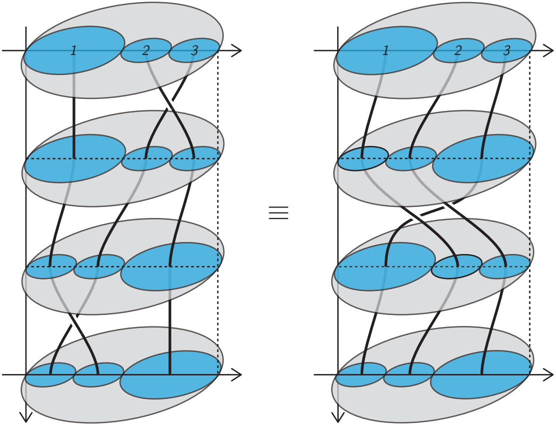


- and we have two hexagon equations combining associators and braidings.

- The first one reads:



- and the second one reads:



Theorem (Mac Lane + Joyal-Street): *An operad morphism* $\phi : \mathsf{PaB} \to \mathsf{P}$, *where* P *is any operad in the category of categories, is uniquely determined by:*

- *an object* $m \in \mathtt{Ob}\,\mathsf{P}(2)$, *which represents the image of the little 2-disc configuration* $\mu \in \mathsf{D}_2(2)$ *under the map* $\phi : \mathtt{Ob}\,\mathsf{PaB}(2) \to \mathtt{Ob}\,\mathsf{P}(2)$,
- *an isomorphism* $a \in \mathtt{Mor}_{\mathsf{P}(3)}(m \circ_1 m, m \circ_2 m)$, *which represents the image of the associator* $\alpha \in \mathtt{Mor}_{\mathsf{PaB}(3)}(\mu \circ_1 \mu, \mu \circ_2 \mu)$,
- *and an isomorphism* $c \in \mathtt{Mor}_{\mathsf{P}(2)}(m, tm)$, *which represents the image of the braiding* $\tau \in \mathtt{Mor}_{\mathsf{PaB}(2)}(\mu, t\mu)$,
- *such that* a *and* c *satisfy the analogue of the usual pentagon and hexagon relations of braided monoidal categories in* $\mathtt{Mor}\,\mathsf{P}$.

This result implies that PaB is generated by operations defining the structure of a braided monoidal category.

Appendix: The rational homotopy theory interpretation of Drinfeld's associators

Construction:

- Take the groups of group like elements

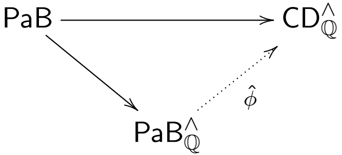
$$\mathbb{G}(\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)) = \{u \in \hat{\mathbb{U}}(\hat{\mathfrak{p}}(r)) | \, \epsilon(u) = 1\,,\, \Delta(u) = u \hat{\otimes} u\}$$

  in the complete Hopf algebras $\hat{\mathbb{U}}(\hat{\mathfrak{p}}(r))$.
- Regard these groups as the morphism sets of groupoids $\mathrm{CD}_{\mathbb{Q}}^{\wedge}(r)$ such that $\mathtt{Ob}\,\mathrm{CD}_{\mathbb{Q}}^{\wedge}(r) = \mathrm{pt}$.
- These groupoids $\mathrm{CD}_{\mathbb{Q}}^{\wedge}(r)$ form an operad (in the category of groupoids) $\mathrm{CD}_{\mathbb{Q}}^{\wedge}$, with the composition products on morphisms

$$\underbrace{\mathtt{Mor}\,\mathrm{CD}_{\mathbb{Q}}^{\wedge}(k)}_{=\mathbb{G}\,\hat{\mathbb{U}}\hat{\mathfrak{p}}(k)} \times \underbrace{\mathtt{Mor}\,\mathrm{CD}_{\mathbb{Q}}^{\wedge}(l)}_{=\mathbb{G}\,\hat{\mathbb{U}}\hat{\mathfrak{p}}(l)} \xrightarrow{\circ_i} \underbrace{\mathtt{Mor}\,\mathrm{CD}_{\mathbb{Q}}^{\wedge}(k+l-1)}_{=\mathbb{G}\,\hat{\mathbb{U}}\hat{\mathfrak{p}}(k+l-1)}$$

  induced by the operadic composition of chord diagrams.

- **Reminder:** We have $\pi\,\mathsf{D}_2 \sim \mathsf{PaB} \Rightarrow \mathsf{D}_2 \sim \mathtt{B}(\mathsf{PaB})$ and $\langle \mathsf{Pois}^c_1(r)\rangle = \mathtt{MC}_\bullet(\hat{\mathfrak{p}}(r)) \sim \mathtt{B}(\mathbb{G}\,\hat{\mathbb{U}}\hat{\mathfrak{p}}(r)) = \mathtt{B}(\mathsf{CD}^\wedge_{\mathbb{Q}}(r))$.
- **Definition:** *The set of Drinfeld's associators* $\mathtt{Ass}_{\mathbb{Q}}$ *is the set of operad morphisms* $\phi : \mathsf{PaB} \to \mathsf{CD}^\wedge_{\mathbb{Q}}$ *whose extension to a Malcev completion of the operad of parenthesized braids*



is an equivalence of groupoids arity-wise.

- **Observation:** Any such $\phi : \mathsf{PaB} \to \mathsf{CD}^\wedge_{\mathbb{Q}}$ gives a rational weak-equivalence on classifying spaces:

$$\phi_* : \mathtt{B}(\mathsf{PaB}) \xrightarrow{\sim^{\mathbb{Q}}} \mathtt{B}(\mathsf{CD}^\wedge_{\mathbb{Q}})$$