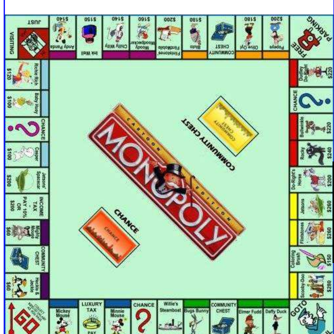
With $\omega\epsilon\beta :=$ http://www.math.toronto.edu/drorbn/Talks/Hamburg-1208

A Quick Introduction to Khovanov Homology

Dror Bar–Natan,
Hamburg, August 2012

Abstract. I will tell the Kauffman bracket story of the Jones polynomial as Kauffman told it in 1987, then the Khovanov homology story as Khovanov told it in 1999, and finally the "local Khovanov homology" story as I understood it in 2003. At the end of our 90 minutes we will understand what is a "Jones homology", how to generalize it to tangles and to cobordisms between tangles, and why it is computable relatively efficiently. But we will say nothing about more modern stuff — the Rasmussen invariant, Alexander and HOMFLYPT knot homologies, and the categorification of sl_2 and other Lie algebras.

Why Bother?



$\omega\epsilon\beta$/webcrafterz

$\omega\epsilon\beta$/onemoretap

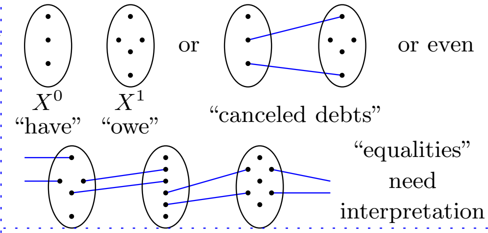
$\omega\epsilon\beta$/cheat-pcgame

What is Categorification=Concretization=de-abstraction? "3" is {cow, cow, cow} and {pig, pig, pig} and many other things...
...categorification is choosing which 3 it is!

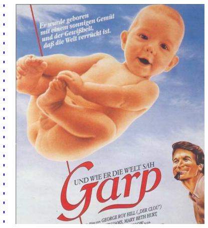
$\mathbb{N}$. Natural numbers $\mapsto$ finite sets, equalities $\mapsto$ bijections, inequalities $\mapsto$ injections and surjections:

$$\binom{2n}{n} = \sum \binom{n}{k}^2 \mapsto$$
$$\binom{X\times\{1,2\}}{|X|} \leftrightarrow \bigcup \binom{X}{k}\times\binom{X}{k}.$$

$\mathbb{Z}$. Negative numbers:



The Philosophy Corner

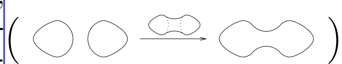


Weaker Categorification. Do the same in the category of vector spaces: "3" becomes V s.t. $\dim V = 3$, or better, $V^\bullet = (\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$ and $\chi(V^\bullet) := \sum(-1)^r\dim V^r = 3 = \sum(-1)^r\dim H^r$. Equalities become homotopies between complexes.

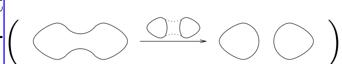
Khovanov: $K(L)$ is a chain complex of graded $\mathbb{Z}$-modules;
$V=\text{span}\langle v_+,v_-\rangle$; $\deg v_\pm=\pm 1$; $\text{qdim}\,V = q+q^{-1}$;

$$K(\bigcirc^k) = V^{\otimes k};\quad K(\times) = \text{Flatten}\left(0\to \underset{\text{height }0}{K()(){1}} \to \underset{\text{height }1}{K(\asymp)\{2\}} \to 0\right);$$

$$K(\times) = \text{Flatten}\left(0\to \underset{\text{height }-1}{K(\asymp)\{-2\}} \to \underset{\text{height }0}{K()()\{-1\}} \to 0\right);$$



$\longrightarrow (V\otimes V \xrightarrow{m} V)$ $\quad m: \begin{cases} v_+\otimes v_-\mapsto v_- & v_+\otimes v_+\mapsto v_+\\ v_-\otimes v_+\mapsto v_- & v_-\otimes v_-\mapsto 0\end{cases}$

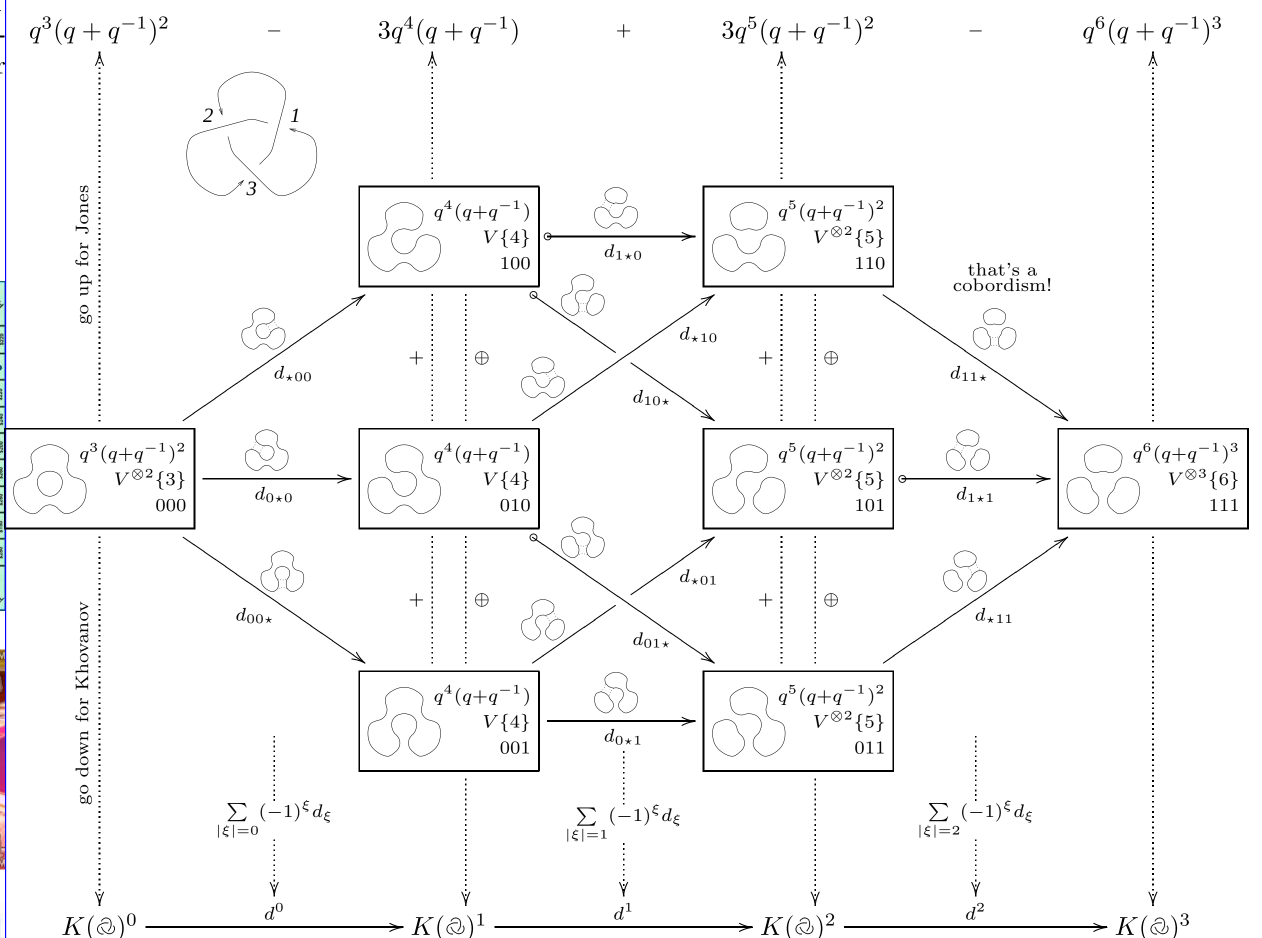


$\longrightarrow (V \xrightarrow{\Delta} V\otimes V)$ $\quad \Delta: \begin{cases} v_+\mapsto v_+\otimes v_- + v_-\otimes v_+\\ v_-\mapsto v_-\otimes v_-\end{cases}$

Categorifying $\mathbb{Z}[q^{\pm1}]$. $f=\sum a_j q^j$ becomes $V=\bigoplus V_j$ s.t. $\text{qdim}\,V := \sum q^j\dim V_j = f$, or better, $V^\bullet = (\cdots V^{r-1}\to V^r\to V^{r+1}\cdots)$ s.t. $d^2=0$, $\deg d = 0$, and $\chi_q(V^\bullet) := \sum(-1)^r\,\text{qdim}\,V^r = f = \sum(-1)^r\,\text{qdim}\,H^r$.

Note. Setting $V\{l\}_j := V_{j-l}$, we get $\text{qdim}\,V\{l\} = q^l\,\text{qdim}\,V$.

Example: $\qquad = \quad q+q^3+q^5-q^9.$



(here $(-1)^\xi := (-1)^{\sum_{i<j}\xi_i}$ if $\xi_j=\star$) $\qquad = \quad K(\text{trefoil}).$

Theorem 1. The graded Euler characteristic of $K(L)$ is $J(L)$.

Theorem 2. The homology $\text{Kh}(L)$ of $K(L)$ is a link invariant.

Theorem 3. $\text{Kh}(L)$ is strictly stronger than $J(L)$: $J(\bar 5_1) = J(10_{132})$ yet $\text{Kh}(\bar 5_1)\neq\text{Kh}(10_{132})$.

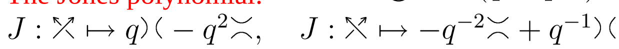
References. Khovanov's arXiv:math.QA/9908171 and arXiv:math.QA/0103190 and my
http://www.math.toronto.edu/~drorbn/papers/Categorification/.

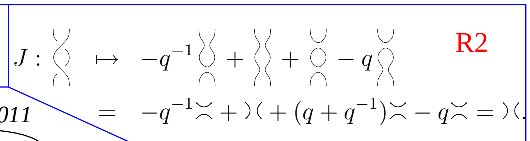
Dror Bar-Natan: Talks: Hamburg-1208

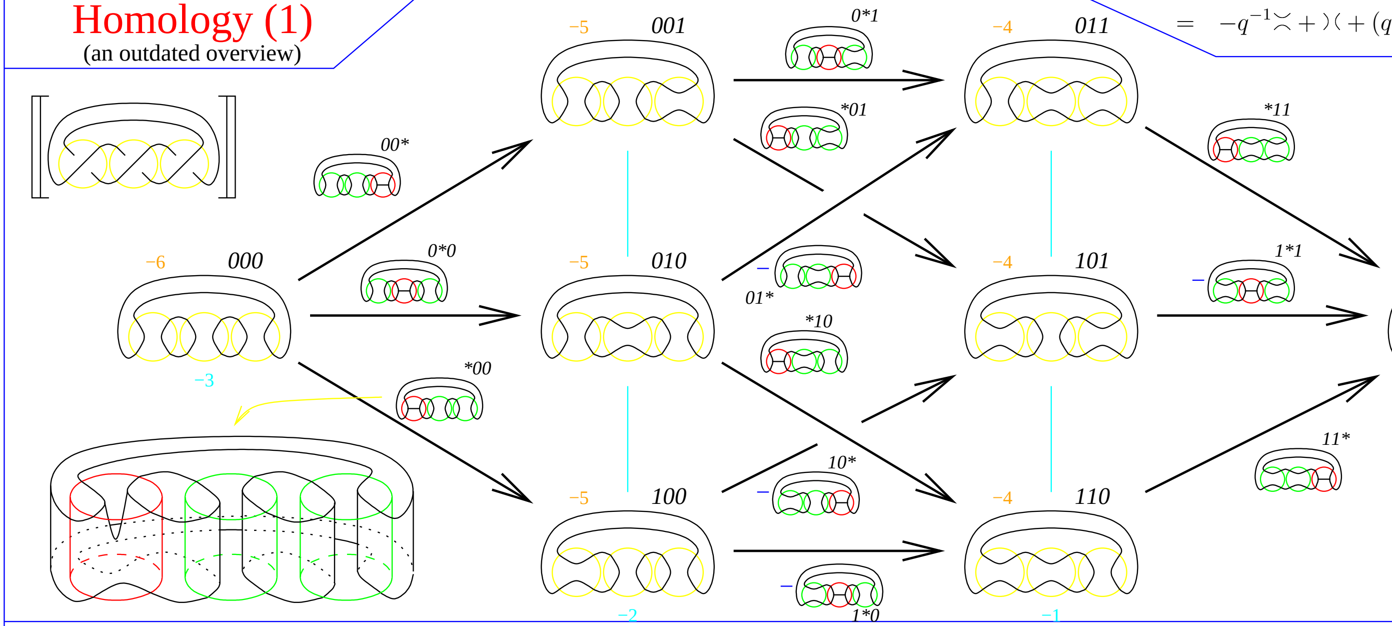
Local Khovanov Homology (1)

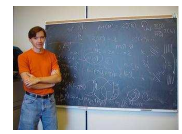
(an outdated overview)

The Jones polynomial: $\bigcirc^k \mapsto (q+q^{-1})^k$



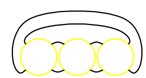
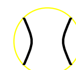
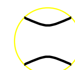


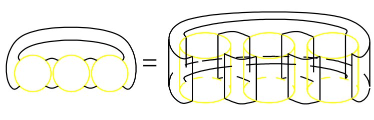
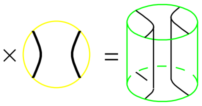
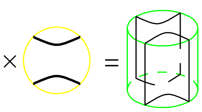
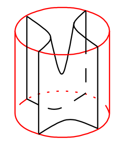




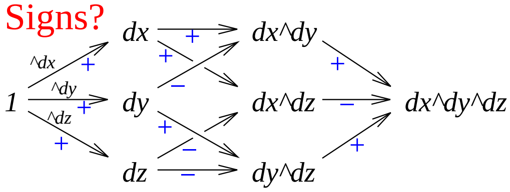
Mikhail Khovanov

What is it?

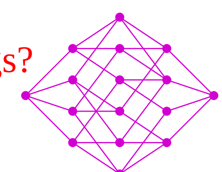
A cube for each knot/link projection; Vertices: All fillings of  with  or with .

Edges: All fillings of $I\times$  with $I\times$  or with $I\times$  and precisely one .

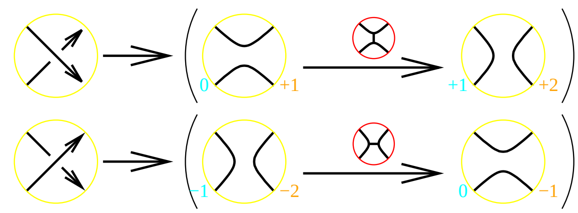
Signs?



More crossings?



General Crossings



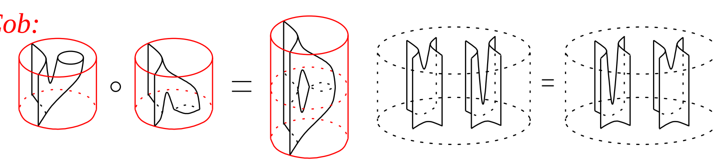
Where does it live?

In *Kom(Mat(<Cob> / {S, T, G, NC})) / homotopy*

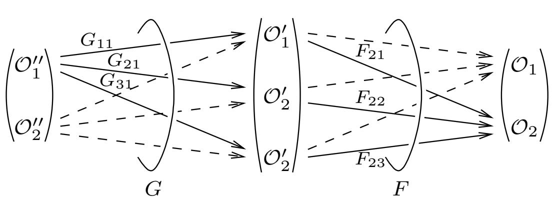
Kom: Complexes Mat: Matrices

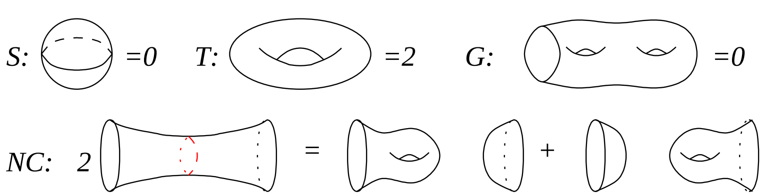
Cob: Cobordisms <...>: Formal lin. comb.

Cob:



Mat(C):





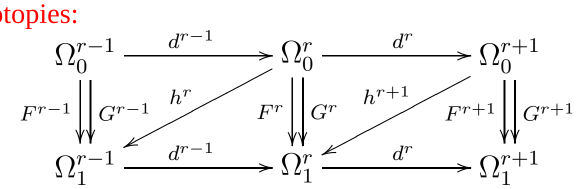
Complexes:

$$\Omega = \left(\Omega^{-n_-} \longrightarrow \Omega^{-n_-+1} \longrightarrow \cdots \qquad \longrightarrow \Omega^{n_+}\right)$$

Morphisms:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \Omega_0^{r-1} & \xrightarrow{d^{r-1}} & \Omega_0^r & \xrightarrow{d^r} & \Omega_0^{r+1} & \longrightarrow \cdots \\ & \downarrow F^{r-1} & & \downarrow F^r & & \downarrow F^{r+1} & \\ \cdots \longrightarrow & \Omega_1^{r-1} & \xrightarrow{d^{r-1}} & \Omega_1^r & \xrightarrow{d^r} & \Omega_1^{r+1} & \longrightarrow \cdots \end{array}$$

Homotopies:



$$F^r - G^r = h^{r+1}d^r + d^{r-1}h^r$$

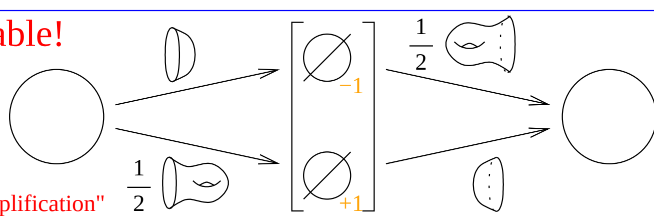
All arrows in an arbitrary additive category!

The Main Point. "The cube", $\mathrm{Kh}(L)$, is an up-to-homotopy invariant of knots and links. It's Euler characteristic is the Jones polynomial, yet it is strictly stronger than the Jones polynomial. It is functorial (in the appropriate sense) and practically computable.

The Categorification Speculative Paradigm. • Every object in math is the Euler characteristic of a complex.
• Every operation lifts to an operation between complexes.
• Every identity remains true, up to homotopy.

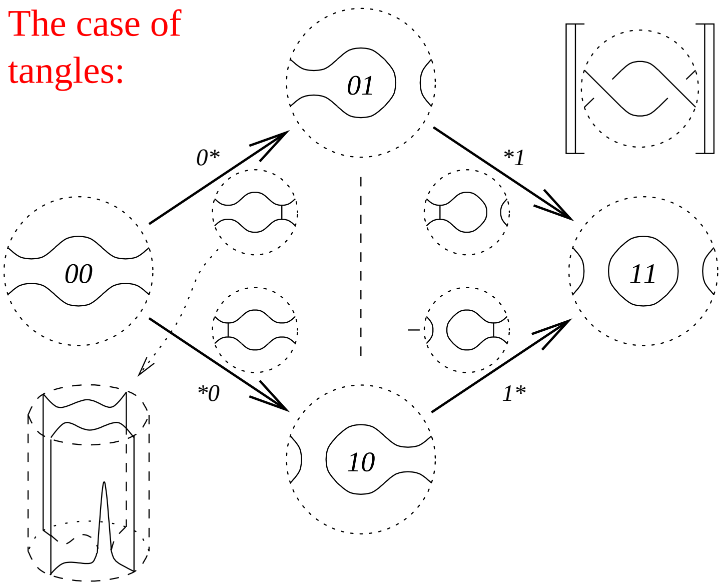
Computable!

via "complex simplification"



Local Khovanov Homology (2)

The case of tangles:



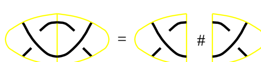
The Reduction Lemma. If ϕ is an isomorphism then the complex

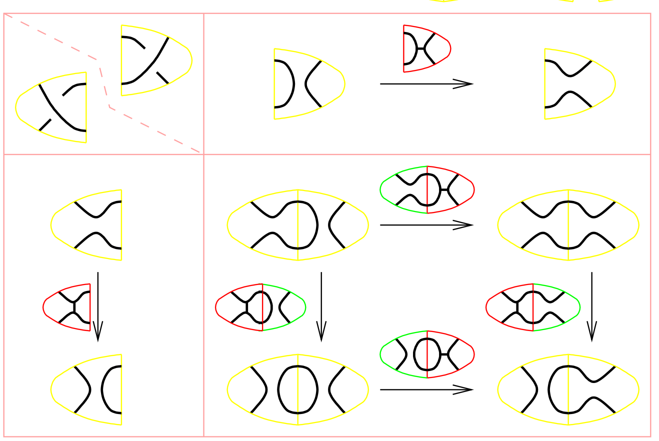
$$[C] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} [F]$$

is isomorphic to the (direct sum) complex

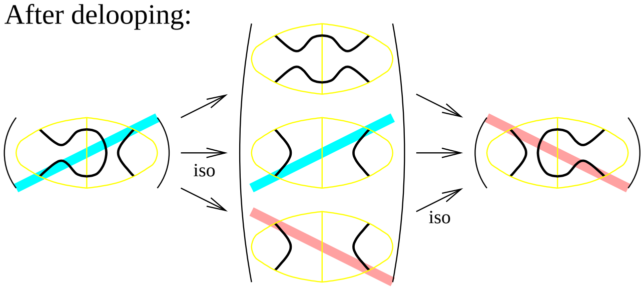
$$[C] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix} b_1 \\ D \end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix} b_2 \\ E \end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} [F]$$

Invariance under R2.



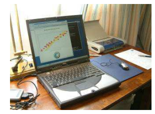


After delooping:

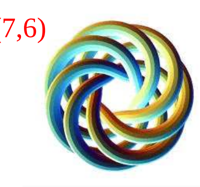


Kurt Reidemeister

I mean business.

In 1 day  says

T(7,6)



$\dim_j H_r$ is given by:

j \ r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
57																	1			1
55																	1	1		
53															1	2		1	1	
51													1	1		2	1			
49														3	1		1			
47												3	1		1					
45										2		1	2							
43								1		1	2									
41						1		1	2		1									
39						1	1		1											
37				1	1		1													
35					1															
33			1																	
31	1																			
29	1																			

the alternating knots zone

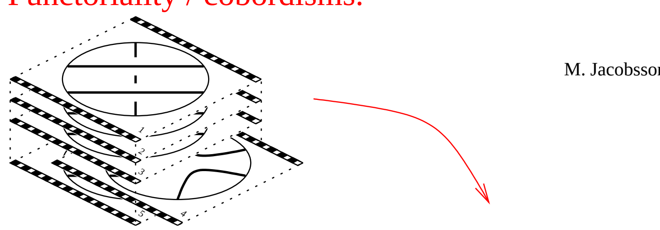
Old techniques:

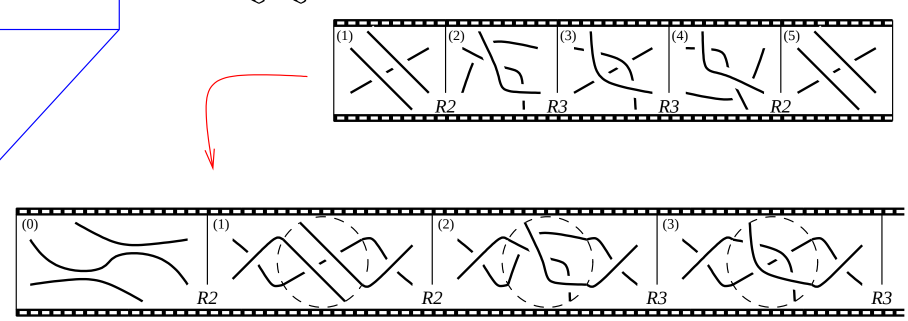
~1,000 years,
~1GGb RAM.

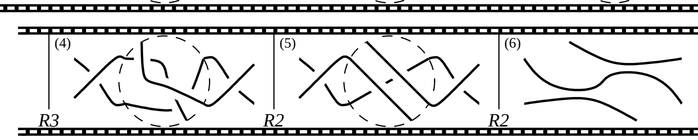
(now down to seconds)

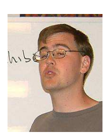
Functoriality / cobordisms.

M. Jacobsson





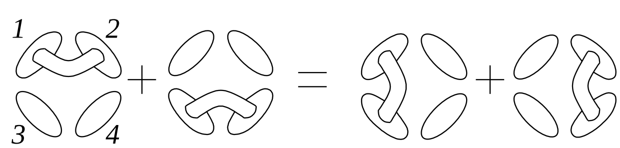




J. Rasmussen: Leads to a no-analysis proof of a conjecture by Milnor.

A more general theory:

Remove G and NC, add

4Tu: 

(minor further revisions are necessary)

"God created the knots,

all else in topology is the work of mortals"

Leopold Kronecker (modified)

Visit!

Edit!

http://katlas.org

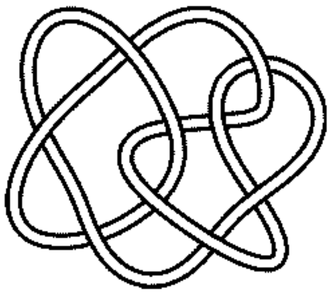
http://www.math.toronto.edu/~drorbn/papers/Cobordism/
http://www.math.toronto.edu/~drorbn/papers/FastKh/
http://www.math.toronto.edu/~drorbn/Talks/Hamburg-1208/

The Most Important Missing Infrastructure Project in Knot Theory

January-23-12
10:12 AM

An "infrastructure project" is hard (and sometimes non-glorious) work that's done now and pays off later.

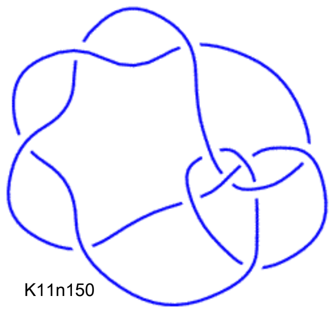
An example, and the most important one within knot theory, is the tabulation of knots up to 10 crossings. I think it precedes Rolfsen, yet the result is often called "the Rolfsen Table of Knots", as it is famously printed as an appendix to the famous book by Rolfsen. There is no doubt the production of the Rolfsen table was hard and non-glorious. Yet its impact was and is tremendous. Every new thought in knot theory is tested against the Rolfsen table, and it is hard to find a paper in knot theory that doesn't refer to the Rolfsen table in one way or another.



(KnotPlot image)

9_42 is Alexander Stoimenow's favourite

A second example is the Hoste-Thistlethwaite tabulation of knots with up to 17 crossings. Perhaps more fun to do as the real hard work was delegated to a machine, yet hard it certainly was: a proof is in the fact that nobody so far had tried to replicate their work, not even to a smaller crossing number. Yet again, it is hard to overestimate the value of that project: in many ways the Rolfsen table is "not yet generic", and many phenomena that appear to be rare when looking at the Rolfsen table become the rule when the view is expanded. Likewise, other phenomena only appear for the first time when looking at higher crossing numbers.



(Knotscape image)

But as I like to say, knots are the wrong object to study in knot theory. Let me quote (with some variation) my own (with Dancso) "WKO" paper:

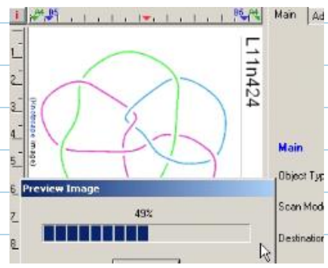
> Studying knots on their own is the parallel of studying cakes and pastries as they come out of the bakery - we sure want to make them our own, but the theory of desserts is more about the ingredients and how they are put together than about the end products. In algebraic knot theory this reflects through the fact that knots are not finitely generated in any sense (hence they must be made of some more basic ingredients), and through the fact that there are very few operations defined on knots (connected sums and satellite operations being the main exceptions), and thus most interesting properties of knots are transcendental, or non-algebraic, when viewed from within the algebra of knots and operations on knots (see [AKT-CFA]).
>
> The right objects for study in knot theory are thus the ingredients that make up knots and that permit a richer algebraic structure. These are braids (which are already well-studied and tabulated) and even more so tangles and tangled graphs.

The interchange of I-95 and I-695, northeast of Baltimore. (more)

Thus in my mind the most important missing infrastructure project in knot theory is the tabulation of tangles to as high a crossing number as practical. This will enable a great amount of testing and experimentation for which the grounds are now still missing. The existence of such a tabulation will greatly impact the direction of knot theory, as many tangle theories and issues that are now ignored for the lack of scope, will suddenly become alive and relevant. The overall influence of such a tabulation, if done right, will be comparable to the influence of the Rolfsen table.

From [AKT-CFA]

Aside. What are tangles? Are they embedded in a disk? A ball? Do they have an "up side" and a "down side"? Are the strands oriented? Do we mod out by some symmetries or figure out the action of some symmetries? Shouldn't we also calculate the affect of various tangle operations (strand doubling and deletion, juxtapositions, etc.)? Shouldn't we also enumerate virtual tangles? w-tangles? Tangled graphs?



From [FastKh]

In my mind it would be better to leave these questions to the tabulator. Anything is better than nothing, yet good tabulators would try to tabulate the more general things from which the more special ones can be sieved relatively easily, and would see that their programs already contain all that would be easy to implement within their frameworks. Counting legs is easy and can be left to the end user. Determining symmetries is better done along with the enumeration itself, and so it should.

An even better tabulation should come with a modern front-end - a set of programs for basic manipulations of tangles, and a web-based "tangle atlas" for an even easier access.

http://katlas.org/

Overall this would be a major project, well worthy of your time.

(Source: http://katlas.math.toronto.edu/drorbn/AcademicPensieve/2012-01/)