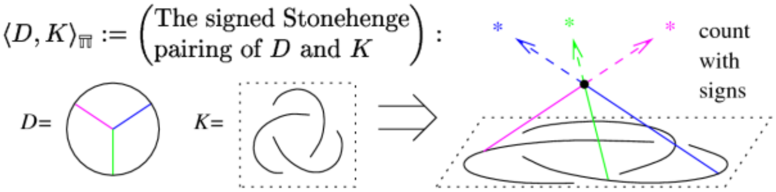
Lecture 5 Extras

Review Material (mostly)

Dror Bar-Natan at Villa de Leyva, July 2011, http://www.math.toronto.edu/~drorbn/Talks/Colombia-1107

$\langle D, K\rangle_{\text{ㅠ}} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}$: count with signs



Thus we consider the generating function of all stellar coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{\text{3-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\text{ㅠ}} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

Theorem. Modulo Relations, $Z(K)$ is a knot invariant!

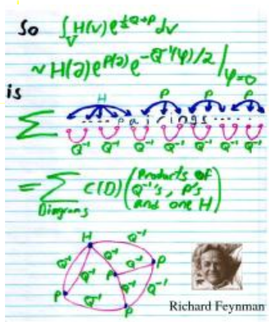
When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,

An intersection line cuts through the knot – Solution: Impose STU,

The Gauss curve slides over a star – Solution: Multiply by

(see below) (similar argument)

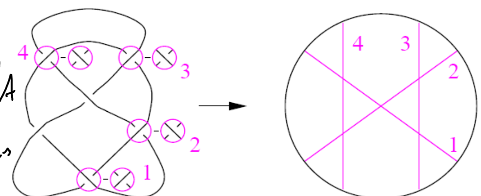


It all is perturbative Chern–Simons–Witten theory:

$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right]$$

$$\to \sum_{\substack{D:\text{ Feynman}\\ \text{diagram}}} W_{\mathfrak{g}}(D) \sum \mathcal{E}(D) \to \sum_{\substack{D:\text{ Feynman}\\ \text{diagram}}} D \sum \mathcal{E}(D)$$

Definition. Any $V:\{\text{knots}\}\to$ Abelian Group A can be extended to "knots w/ double points"



using $V(\times) = V(\nearrow) - V(\nwarrow)$. (Think "differentiation")

Definition. V is of type m if always

$$V(\underbrace{\times\times\cdots\times}_{m+1}) = 0 \quad \text{(think "polynomial")}$$

Conjecture. Finite type invariants separate knots.

Theorem. If $C(K) = \sum_{m=0}^{\infty} V_m(K) z^m$ then V_m is of type m.

Proof. $C(\times) = C(\nearrow) - C(\nwarrow) = zC(\uparrow\uparrow)$

Proposition. The fundamental thm holds iff there exists an expansion

$Z: K \to \hat{\mathcal{A}}$ s.t. if K is m-singular, then

$$Z(K) = D_K + \text{higher degrees}$$

The big picture, "u" case.

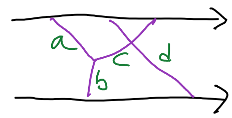
Topology — Combinatorics

$K \xrightarrow[\text{expansion}]{Z} \mathcal{A} \xrightarrow{\text{choose } \mathfrak{g}, R} \mathbb{Q}$; $\mathcal{A} \xrightarrow{\text{choose } \mathfrak{g}} U(\mathfrak{g})$

$\{\text{knots etc.}\}$ $\quad$ $\{\text{chord diagrams}\}/\text{IHX, STU, 4T}$

high algebra — low algebra

Low algebra. $\mathcal{A}(\uparrow\uparrow) \to U(\mathfrak{g})^{\otimes 2}$ via

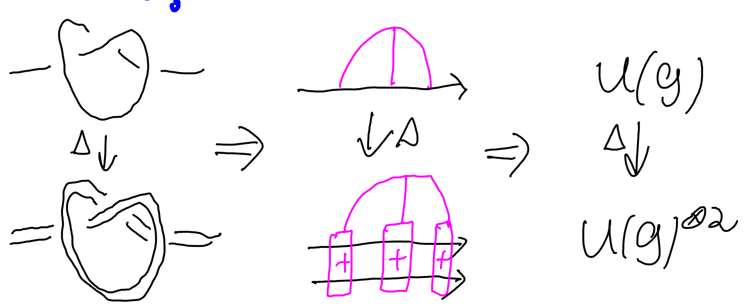


$$\longrightarrow \sum_{a\text{–}d} f_{abc} \begin{pmatrix} x_a x_d x_c \\ \otimes \\ x_b x_d \end{pmatrix}$$

& likewise, $\mathcal{A}(\uparrow_n) \to U(\mathfrak{g})^{\otimes n} \Rightarrow$

$\mathcal{A}(\uparrow_n)$ is "universal universal rep. theory"!

What's Δ?



A "Homomorphic Expansion" $Z: K \to \mathcal{A}$ is an expansion that intertwines all relevant algebraic ops. If K is finitely presented, finding Z is **High Algebra.**

An Associator:

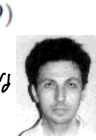
Quantum Algebra's "root object"

$$(AB)C \xrightarrow{\Phi\in\mathcal{U}(\mathfrak{g})^{\otimes 3}} A(BC)$$

satisfying the "pentagon",

$$\begin{array}{ccc} ((AB)C)D & \longrightarrow & (AB)(CD) \\ & (\Delta 11)\Phi & \\ \Phi 1 \swarrow & & \searrow (11\Delta)\Phi \\ (A(BC))D & & A(B(CD)) \\ (1\Delta 1)\Phi \searrow & & \nearrow 1\Phi \\ & A((BC)D) & \end{array}$$

$$\Phi 1\cdot(1\Delta 1)\Phi\cdot 1\Phi = (\Delta 11)\Phi\cdot(11\Delta)\Phi$$



The hexagon? Never heard of it.

See Also. B-N & Dancso, arXiv: 1103.1896