From Stonehenge to Witten Skipping all the Details

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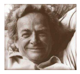
Dror Bar–Natan, University of Toronto

Justin Sawon told us about Feynman diagrams for the Chern–Simons–Witten theory:



$$\int_{\mathfrak{g}\text{-connections}} \mathcal{D}A\, hol_K(A) \exp\left[\frac{ik}{4\pi}\int_{\mathbb{R}^3} \operatorname{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\right] \longrightarrow \sum_{\text{Feynman Diagrams } D} W_{\mathfrak{g}}(D) \not\sum \mathcal{E}(D) \longrightarrow \sum_{\text{Feynman Diagrams } D} D \not\sum \mathcal{E}(D)$$

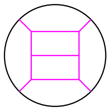
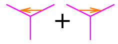
When all the dust settles this becomes the generating function of all stellar coincidences:

$$Z(K) := \lim_{N\to\infty} \sum_{3\text{-valent } D} \frac{1}{2^c c! \binom{N}{e}} \langle D, K\rangle_{\overline{\mathrm{m}}} D \cdot \begin{pmatrix}\text{framing-}\\ \text{dependent}\\ \text{counter-term}\end{pmatrix} \in \mathcal{A}(\circlearrowleft)$$

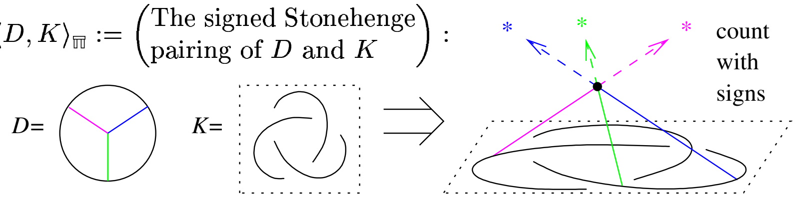
N := # of stars

c := # of chopsticks

e := # of edges of D

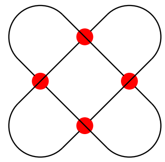
$\mathcal{A}(\circlearrowleft)$:=Span $\langle$  $\rangle$ / oriented vertices, AS:  =0, & more relations

$\langle D, K\rangle_{\overline{\mathrm{m}}} := \begin{pmatrix}\text{The signed Stonehenge}\\ \text{pairing of } D \text{ and } K\end{pmatrix}$:



The Gaussian linking number

$$lk(\bigcirc\!\!\bigcirc) = \frac{1}{2}\sum_{\substack{\text{vertical}\\ \text{chopsticks}}} (\text{signs})$$

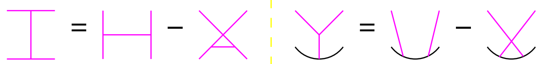


When deforming, catastrophes occur when:

A plane moves over an intersection point – Solution: Impose IHX,

An intersection line cuts through the knot – Solution: Impose STU,

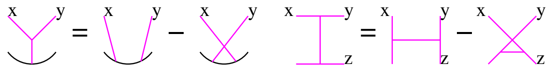
The Gauss curve slides over a star – Solution: Multiply by a framing–dependent counter–term.



(see below) (similar argument) (not shown here)

Theorem. Modulo Relations, *Z(K)* is a knot invariant!

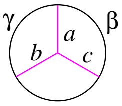
Related to Lie algebras



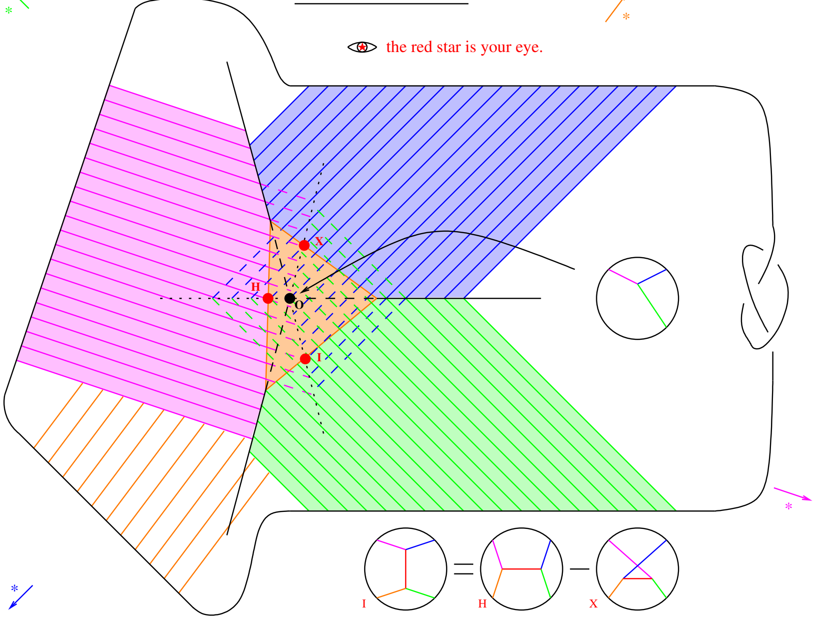
More precisely, let $\mathfrak{g} = \langle X_a \rangle$ be a Lie algebra with an orthonormal basis, and let $R = \langle v_\alpha \rangle$ be a representation. Set

$$f_{abc} := \langle [a,b], c\rangle \qquad X_a v_\beta = \sum_\beta r^\beta_{a\gamma} v_\gamma$$

and then

$W_{\mathfrak{g},R}$:  $\longrightarrow \displaystyle\sum_{abc\alpha\beta\gamma} f_{abc} r^\beta_{a\gamma} r^\gamma_{b\alpha} r^\alpha_{c\beta}$

The IHX Relation



$W_{\mathfrak{g},R} \circ Z$ is often interesting:

$\mathfrak{g} = sl(2)$ →  The Jones polynomial

$\mathfrak{g} = sl(N)$ →  The HOMFLYPT polynomial

$\mathfrak{g} = so(N)$ →  The Kauffman polynomial

Sorry, there's space left and I run out of things to say

"God created the knots, all else in topology is the work of man."

Leopold Kronecker

(modified)

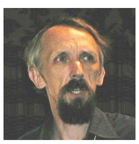
Definition. V is finite type (Vassiliev, Goussarov) if it vanishes on sufficiently large alternations as on the right

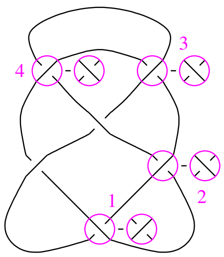
Theorem. All knot polynomials (Conway, Jones, etc.) are of finite type.

Conjecture. (Taylor's theorem) Finite type invariants separate knots.

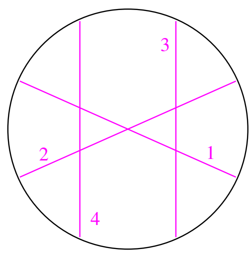
Theorem. *Z(K)* is a universal finite type invariant!

(sketch: to dance in many parties, you need many feet).





The Miller Institute knot



This handout is at http://www.math.toronto.edu/~drorbn/Talks/BIRS−0311