Dror Bar-Natan — Research Proposal

(web version: http://www.math.toronto.edu/~drorbn/ResearchProposal/)
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Background on Finite Type Invariants

A *knot* is a circular piece of string placed in space, freely allowed to move but never to cross itself. A *link* is much the same, except we can place several circular pieces of string rather than just one. A *knotted trivalent graph* is again similar, though now we may solder together strings to get vertices in which the ends of up to three open strings are attached. I will refer to these types of objects commonly as *knotted objects*. The collection of all knotted objects is quite unwieldy, and given two, it is often difficult to tell if they are the same or not. Thus an *invariant* of knotted objects is simply a function from the set of knotted objects to some simpler set in which equality is easier to test. Of course, "better" invariants are valued in "richer" sets which allow one to read more about the original knotted objects from the value of its invariants.

Over the last 10-12 years one such invariant Z (often called "the Kontsevich integral" after its first definition) attracted a lot of attention. The invariant Z is valued in a certain space $\mathcal{A}$ of formal linear combinations of trivalent graphs (unknotted, plain and easy trivalent graphs, distinct from the ones in the domain of Z) modulo certain relations that relate graphs that differ only in some local way: the IHX relation $I = H - X$, the STU relation $S = T - U$, etc. Here are some of the reasons why Z is so interesting:

1. There is a natural class of invariants of knotted objects, called *finite type invariants*, and Z is universal in that class. In detail: Every invariant V can be extended to be defined on knotted objects that are allowed to have a finite number of self intersections by recursively using the local formula $V^{(m)}(\times) = V^{(m-1)}(\times) - V^{(m-1)}(\times)$, where m is the number of self intersections. Differences are relatives of derivatives, and hence the extended invariant $V^{(m)}$ may well be thought of as the mth derivative of V. An invariant V is said to be of finite type, if, like a polynomial, one of its high derivatives $V^{(m)}$ is identically equal to 0. It turns out that Z is universal in this natural class of finite type invariants — every finite type invariant factors through Z and Z can be reconstructed given a complete knowledge of finite type invariants. See more at [B2].

2. As many of the previously known knot invariants factor through finite type invariants it follows that invariants such as the Alexander-Conway polynomial, the Jones polynomial, the HOMFLY and Kauffman polynomials and Reshetikhin-Turaev invariants can all be reconstructed from knowledge of Z. See more at [B2].

3. The target space $\mathcal{A}$ of Z is closely related to Lie algebras [B2]. Thus much of the rich structure of Lie algebras can be translated to $\mathcal{A}$ terms. This has implications in both directions — using tools borrowed from the theory of Lie algebras we can learn things

about $\mathcal{A}$, and more surprisingly, using knot theory we can learn some things about Lie algebras (here I am referring for example to the recently discovered explanation of the Harish-Chandra Duflo isomorphism of the theory of Lie algebras in terms of a knot theoretic version of the equality $1 + 1 = 2$, see [BLT]).

4. The original definition of Z by Kontsevich relates Z to the Knizhnik-Zamolodchikov equation and hence to conformal field theory and statistical mechanics.

5. Perhaps the nicest definition of Z is using the Chern-Simons-Witten (CSW) path integral and Feynman diagrams [B1]. One may attempt to compute the large k asymptotics of the latter path integral over the space of connections A on $\mathbb{R}^3$,

$$\int_A \mathcal{D}A\,\mathcal{O}(K)\exp\left(\frac{ik}{4\pi}\int_{\mathbb{R}^3}\mathrm{tr}(A\wedge dA+\frac{2}{3}A\wedge A\wedge A)\right),$$

   using Feynman diagrams. After some repackaging, the end result is a linear combinations of graphs such as the ones making $\mathcal{A}$, with coefficients given by some complicated integrals. This result is our invariant Z, perhaps up to some renormalization.

6. The hairy integrals of the previous point can be reinterpreted as computations of degrees of certain maps of configuration spaces of points in $\mathbb{R}^3$ into various products of spheres (see e.g. [BT]) and again as a beautiful discrete counting problem of "tinkertoy diagrams" [T] or "chopstick towers" [B6].

7. There are algebraic approaches to the computation of Z: One finds some algebraic context within which the set of knotted objects is finitely presented using finitely many operations and finitely many generators and relations. This done, it is now enough to specify how Z should behave under the operations and to make "good" guesses for the values of Z on the generators, good enough so that the relations will be satisfied. Several such approaches exist:

   (a) Using parenthesized tangles, the computation of Z reduces to essentially just one guess, for the value Φ of Z on the associativity morphism. It turns out that the required Φ is essentially a Drinfel'd Associator, and thus its existence (and proper behavior, in several senses) can be deduced from Drinfel'd's work on quasi-triangular quasi-Hopf algebras. See [LM, B3, B5].

   (b) Staying within the context of knotted trivalent graphs it turns out that it is enough to guess the value of Z on the unknotted tetrahedron. This value turns out to be nearly equivalent to an associator Φ. It is also related to quantum $6j$ symbols for arbitrary quantum groups.

   (c) There is an algebraic evaluation of Z along similar lines but using V. Jones' [Jo] notion of planar algebras.

8. Explicit formulas for the values of Z on specific knotted objects are surprisingly difficult to obtain. In the few cases where such values were computed, the computations tend to be intricate but also elegant and inspiring. See [BL, BLT].

9. Finally (though only because the space is short), via a procedure discovered by Le, Murakami and Ohtsuki [LMO] or using the Århus integral of [BGRT], Z can be used as the seed for a construction of a universal finite type invariant of 3-dimensional manifolds.

My Proposed Research

Most of the picture sketched above is understood quite well, but there are still several significant missing pieces:

- Is the invariant coming from the CSW theory precisely equal to the Z defined using the Kontsevich integral, or is the renormalization required really non-trivial? This is the "vanishing of the anomaly" question (see [P1, P2]). I believe I can at least compute the anomaly for a few degrees beyond what is known today, hoping that this will be enough to resolve the question.

- As of now, the CSW construction only works for knots and links, but not for knotted trivalent graphs. This gap is significant because using point 7b we should be able to construct an associator directly from the CSW theory once it will apply to knotted trivalent graphs as well. The problems in extending CSW to knotted graphs are technical in nature and I hope to contribute to their eventual solution.

- In fact, the algebraic theory of point 7b is not yet fully written up. I hope to fix this soon.

- The relationship between point 7b and quantum $6j$ symbols is only half as good as we would like it to be — given $Z(\triangle)$ we can find appropriate solutions of the Biedenharn-Elliot identity, but given solutions of the Biedenharn-Elliot identity we don't know yet how to go back and find $Z(\triangle)$. Given a semisimple Lie algebra $\mathfrak{g}$, one can "see" the entire quotient $U(\mathfrak{g})/\mathfrak{g}$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ using characters of representations. What is missing in our case is a similar theorem about a quotient like $U(\mathfrak{g})^{\otimes n}/\mathfrak{g}$. We hope to state and prove such a theorem.

- It is known that algebraic constructions as in point 7a are stronger than those of point 7c, in the sense that given a construction of the former kind it leads to a construction of the latter kind. It is not known if this dominance is strict. If it isn't, point 7c will become a beautiful new and natural way of arriving at associators. If it is, it means that constructions of the 7c kind are easier than constructions of the 7a kind. That too would be good news for 7a requires associators and the constructions we currently have for associators are far from easy and simple. I hope to clarify these points in my research over the next few years.

- **Last, but perhaps most important** — the picture discussed here is too nice to be buried in hundreds of different publications [B4], and it is time for it to be assembled into a single integrated text. I want to write it!

Background on Categorification

The previous section may give the impression that all there is to know about algebraic knot invariants is in the Kontsevich integral Z. This was nearly true until about two years ago, when Khovanov [K1] proved the following unexpected

Theorem. *(sketch) Given a planar projection of a knot or a link, there is a graded chain complex whose homology is an invariant of the underlying knot or link and whose graded Euler characteristic is the Jones polynomial of that knot or link.*

Thus the Khovanov homology ("categorification") relates to the Jones polynomial like homologies relates to Euler characteristics — potentially, it is vastly richer. (And this homology seems to have nothing to do with $Z\dots$). In the time since Khovanov stated his theorem, this potential seems have to become reality — I have shown [B7] that the Khovanov homology is indeed stronger than the original Jones polynomial, while Khovanov [K3] and Jacobsson [Ja] have shown that "maps between knots" (more precisely — cobordisms between knots) induce invariant maps between their homologies. There are indications that there are parallel categorifications at least of the Alexander polynomial and of the $sl(3)$ invariant of knots and links.

My Proposed Research

My current computer program for computing Khovanov homology is extremely inefficient and the main reason for that is inherently mathematical — as it is, Khovanov's chain complex is just too big. An indication for that is the fact that the rank of the homology is invariably much smaller than the dimensions of the spaces of chains involved. I believe I can do a lot better by mixing some homological algebra and some sophisticated programming, and I hope to do so sometime over the grant period.

The Jacobsson-Khovanov invariant of knot cobordisms was never computed for anything. It may be of huge value, or it may be trivial. With some effort I believe I should be able to compute this invariant on a large number of specific cobordisms and hopefully determine its value.

Very little is known about the potential categorifications of the Alexander polynomial and of the $sl(3)$ invariant. I plan to attempt to find combinatorial constructions for these invariants and to use those constructions for concrete computations.

Computations

I am not sure if a grant proposal is the appropriate forum to come out of the closet, but here I go. I love to write little programs that do mathematically significant things. I've done that all along and I will do that further on. Several of the projects mentioned above are about such programs or will require such programs, and I have several other such projects in mind. So I plan to invest some time to decide on the appropriate foundations: How do I cleanly and elegantly represent knots? Display knots? Manipulate knots? I have written programs that do these things on an ad hoc basis, but my programs don't talk to each other well and cannot form a consistent foundation for further development, so I'll have to start nearly from scratch. And when the foundations are laid, all of knot theory (and especially finite type invariants, graph calculations as in $\mathcal{A}$ and categorification) is there to code. I will need a big computer, students and travel money to visit people with further ideas.

In Summary

My recent progress in research activities related to the proposal. I made significant contributions to almost every topic discussed in this proposal. See the "Contributions" section of my Form 100.
The objectives: both short and long term. As in the "Summary for public release" section of this proposal, my primary goals will be to complete our understanding of the Kontsevich integral of knotted objects and of the Khovanov categorification of certain knot and link invariants, to make these subjects more easily accessible to students and beginning researchers by improving their presentation, and to compute in practice many of the theoretically computable quantities abound in knot theory.
Literature pertinent to the proposal. See the "References" section of this proposal.
Methods and proposed approach. I plan to both work analytically using the traditional mathematical definition-theorem-proof sequence and also to use computers for a large number of different computations.
Anticipated significance of the work. If I'll be able to conclude my book project, it has a chance of being read by many students and beginning researchers. Previous expositions of mine were widely read and a book I would write may have a significant impact. I hope my computations will also be of significance for others, as had been the case with several computational projects I have carried out in the past. As for the analytical research — this is always a wild card — I can only hope it will lead to the eventual completion of our understanding of the relationship between Lie algebras and knot theory and between homological algebra and knot theory.
Training to take place through the proposal. I expect that many parts of my proposed research will be assisted by graduate students and postdoctoral fellows as a part of their training.

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