Problem 1. Let $F : \mathbb{R} \to \mathbb{R}$ be Lipschitz-$L$, that is $|F(x) - F(y)| < L|x - y|$ for all $x, y$. Show that $\text{Var}F(X) \leq L^2\text{Var}X$ for all finite mean random variables $X$.

Problem 2. Let $F : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz-$L$ in all coordinates, let $M_n = F(X_1, \ldots, X_n|\mathcal{F}_k)$ where $\mathcal{F}_k = \sigma(X_1, \ldots X_k)$, and the $X_i$ are independent with finite expectation. Show that

$$
\mathbb{E}[(M_k - M_{k-1})^2|\mathcal{F}_{k-1}] \leq L^2\text{Var}X_k.
$$

Problem 3. Let $M_k$ be a martingale with bounded increments $|M_{k+1} - M_k| \leq b_k$. Show that $Y_k = \exp(M_k - \sum_{i=1}^k b_i^2/2)$ is a supermartingale.

Problem 4. Show that the chromatic number of an ER random graph $G \sim \mathcal{G}(n, p)$ satisfies $P(|\chi(G)) - E\chi(G)| > \gamma\sqrt{n}) \leq c a^\gamma$ for some universal constants $c > 0, a < 1$. 