MOSCOW MATHEMATICAL JOURNAL
Volume 11, Number 2, April–June 2011, Pages 265–283

NEWTON POLYTOPES FOR HOROSPHERICAL SPACES

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To S. M. Gusein-zade for the occasion of his 60th birthday

Abstract. A subgroup H of a reductive group G is horospherical if it contains a maximal unipotent subgroup. We describe the Grothendieck semigroup of invariant subspaces of regular functions on G/H as a semigroup of convex polytopes. From this we obtain a formula for the number of solutions of a system of equations $f_1(x) = \cdots = f_n(x) = 0$ on G/H, where $n = \dim(G/H)$ and each f_i is a generic element from an invariant subspace L_i of regular functions on G/H. The answer is in terms of the mixed volume of polytopes associated to the L_i. This generalizes the Bernstein–Kushnirenko theorem from toric geometry. We also obtain similar results for the intersection numbers of invariant linear systems on G/H.

2000 Math. Subj. Class. 14M17, 14M25.

Key words and phrases. Reductive group, moment polytope, Newton polytope, horospherical variety, Bernstein–Kushnirenko theorem, Grothendieck group.

Introduction

Consider a commutative semigroup S. Two elements a, $b \in S$ are *analogous* and written $a \sim b$ if there is $c \in S$ with $a + c = b + c$ (where we write the semigroup operation additively). This relation is an equivalence relation and respects the addition. The *Grothendieck semigroup* $\mathrm{Gr}(S)$ *of* S is the set of equivalence classes of $\sim$ together with the induced addition. The map which sends an element to its equivalence class is a natural homomorphism $\rho\colon S \to \mathrm{Gr}(S)$. The semigroup $\mathrm{Gr}(S)$ has the *cancelation property*, i.e., if a, b, $c \in \mathrm{Gr}(S)$, the equality $a+c = b+c$ implies $a = b$. Moreover, for any homomorphism $\varphi\colon S \to H$, where H is a semigroup with cancelation, there exists a unique homomorphism $\bar\varphi\colon \mathrm{Gr}(S) \to H$ such that $\varphi = \bar\varphi \circ \rho$. In particular, under the homomorphism φ, analogous elements have the same image. Any semigroup H with cancelation naturally extends to a group, namely its *group of formal differences*. It consists of pairs of elements from H, where two pairs (a, b) and (c, d) are equal if $a + d = b + c$. The *Grothendieck group of a semigroup* S is the group of formal differences of its Grothendieck semigroup $\mathrm{Gr}(S)$.

Received July 14, 2010; in revised form October 18, 2010.

The Grothendieck semigroup of S contains significant information about S and often is more tractable and simpler to describe than S itself.

We will be interested in semigroups of subspaces of functions (as well as sections of line bundles) which arise naturally in algebraic geometry. Let X be an irreducible variety over $\mathbb{C}$ with the field of rational functions $\mathbb{C}(X)$. Consider the collection $\boldsymbol{K}(X)$ of all nonzero finite dimensional subspaces of $\mathbb{C}(X)$. For $L_1, L_2 \in \boldsymbol{K}(X)$ let L_1L_2 denote the linear span of all fg, $f \in L_1$, $g \in L_2$. With this product, $\boldsymbol{K}(X)$ is a commutative semigroup. One shows that for each $L \in \boldsymbol{K}(X)$ there is a largest subspace $\bar{L}$ which is analogous to L called the *completion of* L (see [17, Appendix 4] and [6]).

An interesting and important special case is the algebraic torus $X = (\mathbb{C}^*)^n$. The variety X is a multiplicative group and acts on itself by multiplication. For $\alpha = (a_1, \dots, a_n) \in \mathbb{Z}^n$ let $x^\alpha = x_1^{a_1} \cdots x_n^{a_n}$ denote the corresponding Laurent monomial, which is a regular function on X. Let $A \subset \mathbb{Z}^n$ be a finite subset and let L_A denote the subspace of Laurent polynomials spanned by all the x^α, $\alpha \in A$. The correspondence $A \mapsto L_A$ gives an isomorphism between the semigroup of finite subsets of $\mathbb{Z}^n$ together with the addition of subsets, and the semigroup $\boldsymbol{K}_T(X)$ of invariant subspaces of regular functions on X. One then shows that the Grothendieck semigroup of $\boldsymbol{K}_T(X)$ is isomorphic to the semigroup of integral convex polytopes with the Minkowski sum. Moreover, *for a finite subset A, the completion of the subspace L_A is the subspace $L_{\bar{A}}$, where $\bar{A}$ is the set of all integral points in the convex hull* $\Delta(A)$. From this key fact one can deduce the Bernstein–Kushnirenko theorem: let $A_1, \dots, A_n \subset \mathbb{Z}^n$ be finite subsets. Then the number of solutions in $(\mathbb{C}^*)^n$ of a generic system $f_1(x) = \cdots = f_n(x) = 0$, where $f_i \in L_{A_i}$, is equal to $n!\, V(\Delta_1, \dots, \Delta_n)$. Here, for each i, Δ_i is the convex hull of A_i and V denotes the mixed volume of convex bodies in $\mathbb{R}^n$ (see [11] and [9], also see [12] and [2] for the original papers where this theorem appeared).

In this paper we consider a class of homogeneous spaces of reductive groups which have features similar to the torus $(\mathbb{C}^*)^n$. Let G be a connected reductive algebraic group over $\mathbb{C}$. A subgroup $H \subset G$ is called *horospherical* if it contains a maximal unipotent subgroup of G. The homogeneous space G/H is then called a *horospherical homogeneous space*. The affine embeddings of horospherical homogeneous spaces were studied in [16].

Similar to the case of $(\mathbb{C}^*)^n$, we describe the semigroup of G-invariant subspaces of regular functions on a horospherical homogeneous space $X = G/H$ (respectively its Grothendieck semigroup) in terms of a semigroup of finite subsets (respectively integral convex polytopes). Moreover, we obtain a description of the completion of a finite dimensional G-invariant subspace of regular functions on X. Finally we generalize the above to invariant linear systems on X (Theorem 2.10, Corollary 2.22).

From these we obtain an analogue of the Bernstein–Kushnirenko theorem, that is, a formula for the number of solutions in X of a system $f_1(x) = \cdots = f_n(x) = 0$, where each f_i is a generic element of a finite dimensional G-invariant subspace L_i of regular functions on X. The formula involves certain polytopes associated to the L_i. In fact, we give two answers for the number of solutions: Firstly, we represent it as the mixed integral of an explicitly defined homogeneous polynomial over the so-called *moment polytopes* of the subspaces L_i (Corollary 2.11). Secondly, we construct larger polytopes over the moment polytopes such that their mixed volume is equal to the above mixed integral (Corollary 2.26). We extend all these to the intersection numbers of G-invariant linear systems on X (Corollary 2.23, Corollary 2.27).

This paper is one of a series of papers devoted to the general theory of convex bodies associated to algebraic varieties. In [6] we develop an intersection theory of finite dimensional subspaces of rational functions. In [7] we develop a general theory of Newton-Okounkov bodies associated to algebraic varieties and more generally to graded algebras. Finally in [8] we consider the case of general varieties with a reductive group action.

The results of the present paper are along the same lines as [9]. In there we describe the Grothendieck semigroup of finite dimensional representations of a reductive group G with tensor product. From this we get a proof of Kazarnovskii's theorem on the number of solutions in G of a generic system of equations consisting of matrix elements of representations of G. Some of the background material in the present paper are taken from [9].

Among the different generalizations of the Bernstein–Kushnirenko theorem (e.g., in [3], [10] and [7]), the generalization of the Bernstein–Kushnirenko (for horospherical homogeneous spaces) in this paper is closest to the original Bernstein–Kushnirenko theorem. We expect that other formulae in toric geometry involving Newton polytopes also extend to the horospherical case.

We would like to emphasize that a main difference of our approach (with many other authors) in computation of intersection numbers is that we do not require the varieties to be complete or projective and hence do not need any compactification.

And about the organization of material: Part I is devoted to preliminaries on subspaces of rational functions, linear systems and their intersection indices, notions of mixed volume and mixed integral and finally semigroup of finite subsets of $\mathbb{R}^n$ and its Grothendieck semigroup. In Part II we cover the main results of the paper. Section 2.1 discusses a classification of the horospherical subgroups of G. Section 2.2 describes the semigroup of invariant subspaces, its Grothendieck semigroup and gives a formula for the intersection index on quasi-homogeneous horospherical spaces in terms of moment polytopes. Section 2.3 discusses similar material for invariant linear systems on general horospherical spaces. Finally in Section 2.4 we construct larger polytopes over moment polytopes whose volumes give the intersection index. The last section considers the example of $G = \mathrm{GL}(n, \mathbb{C})$.

Notation. Throughout the paper we will use the following notation.

- G denotes a connected reductive algebraic group over $\mathbb{C}$ with $\dim(G) = d$.
- B denotes a Borel subgroup of G and T and U the maximal torus and maximal unipotent subgroup contained in B respectively. We put $\dim(T) = r$.
- W denotes the Weyl group of (G, T).
- Λ denotes the weight lattice of G (that is, the character group of T), and Λ^+ is the subset of dominant weights (for the choice of B). Put $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Then the convex cone generated by Λ^+ in $\Lambda_{\mathbb{R}}$ is the positive Weyl chamber $\Lambda^+_{\mathbb{R}}$.

- For a weight $\lambda \in \Lambda$, the irreducible G-module corresponding to λ will be denoted by V_λ and a highest weight vector in V_λ will be denoted by v_λ.
- For an algebraic group K, we denote the group of characters of K (written additively) by $\mathfrak{X}(K)$.
- P denotes a parabolic subgroup of G and P' its commutator subgroup.
- H will denote a horospherical subgroup, i.e., a subgroup of G containing a maximal unipotent subgroup U.

1. Preliminaries

1.1. Intersection theory of finite dimensional subspaces and linear systems. Let X be a complex n-dimensional irreducible variety with $\mathbb{C}(X)$ its field of rational functions. Consider the collection $\boldsymbol{K}(X)$ of all nonzero finite dimensional subspaces of $\mathbb{C}(X)$. The *product* of two subspaces $L_1, L_2 \in \boldsymbol{K}(X)$ is the subspace spanned by all the fg where $f \in L_1$, $g \in L_2$. With this product $\boldsymbol{K}(X)$ is a commutative semigroup.

Definition 1.1. The *intersection index* $[L_1, \ldots, L_n]$ is the number of solutions in X of a generic system of equations $f_1 = \cdots = f_n = 0$, where $f_i \in L_i$, $1 \leqslant i \leqslant n$. In counting the solutions, we neglect the solutions x at which all the functions in some space L_i vanish as well as the solutions at which at least one function from some space L_i has a pole.

One shows that the intersection index is well-defined (i.e., is independent of the choice of a generic system) [6]. It is obvious that the intersection index is symmetric with respect to permuting the subspaces L_i. Moreover, the intersection index is linear in each argument. The linearity in first argument means:

$$[L_1'L_1'', L_2, \ldots, L_n] = [L_1', L_2, \ldots, L_n] + [L_1'', L_2, \ldots, L_n], \tag{1}$$

for any $L_1', L_1'', L_2, \ldots, L_n \in \boldsymbol{K}(X)$. From (1) one sees that for a fixed $(n-1)$-tuple of subspaces $L_2, \ldots, L_n \in \boldsymbol{K}(X)$, the map $\pi\colon \boldsymbol{K}(X) \to \mathbb{Z}$ given by $\pi(L) = [L, L_2, \ldots, L_n]$ is a homomorphism from the semigroup $\boldsymbol{K}(X)$ to the additive group of integers. The existence of such a homomorphism shows that the intersection index induces an intersection index on $\mathrm{Gr}(\boldsymbol{K}(X))$, i.e., *the intersection index* $[L_1, \ldots, L_n]$ *remains invariant if we substitute each* L_i *with an analogous subspace* $\tilde{L}_i$.

One can describe the relation of analogous subspaces in a different way as follows (see [6]). A rational function $f \in \mathbb{C}(X)$ is called *integral over the subspace* L if it satisfies an equation

$$f^m + a_1 f^{m-1} + \ldots a_0 = 0$$

with $m > 0$ and $a_i \in L^i$, $1 \leqslant i \leqslant m$. The collection of all the rational functions integral over L forms a finite dimensional subspace $\overline{L}$ called the *completion of* L.

Proposition 1.2. (1) *For any* $L \in \boldsymbol{K}(X)$*, the completion* $\overline{L}$ *belongs to* $\boldsymbol{K}(X)$ *and is analogous to* L. (2) *Moreover, the completion* $\overline{L}$ *contains any subspace* $M \in \boldsymbol{K}(X)$ *analogous to* L. (3) *Two subspaces* $L_1, L_2 \in \boldsymbol{K}(X)$ *are analogous if and only if* $\overline{L}_1 = \overline{L}_2$.

For $L \in \boldsymbol{K}(X)$ define the Hilbert function H_L by $H_L(k) = \dim(\overline{L^k})$. The following theorem provides a way to compute the self-intersection index of a subspace L (see [7, Part II]):

Theorem 1.3. *For any $L \in \boldsymbol{K}(X)$, the limit*

$$a(L) = \lim_{k \to \infty} H_L(k)/k^n$$

exists, and the self-intersection index $[L, \ldots, L]$ is equal to $n!a(L)$.

The proof is based on the Hilbert theorem on the dimension and degree of a subvariety of the projective space.

A linear system on X is a family of effective divisors of the form $D + (f)$, where D is an effective divisor on X and f lies in a finite dimensional subspace $L \subset \mathbb{C}(X)$. In this section we consider the intersection index of linear systems. Let us assume that D is a Cartier divisor and let $\mathcal{L}$ be the line bundle associated to D. Any element $D + (f)$ determines a section of the line bundle $\mathcal{L}$ up to multiplication by a regular nowhere zero function, i.e., an element of $\mathbb{C}[X]^*$. Thus a linear system determines a subspace of holomorphic sections of $\mathcal{L}$, up to multiplication of each section by a function in $\mathbb{C}[X]^*$.

Conversely a finite dimensional subspace E of holomorphic sections $H^0(X, \mathcal{L})$ determines a linear system of divisors $\{\mathrm{Div}(s) \colon 0 \neq s \in E\}$. By abuse of terminology we will refer to $(E, \mathcal{L})$ (or simply E) as a linear system on X. Fix a nonzero section $t \in E$. Then every section $s \in E$ can be written as $s = f_s t$ for a unique $f_s \in \mathbb{C}(X)$. The map $s \mapsto f_s$ identifies E with the subspace of rational functions $\{f_s \colon s \in E\}$.

Let $(E_1, \mathcal{L}_1)$, $(E_2, \mathcal{L}_2)$ be two linear systems on X. There is a tensor product map $H^0(X, \mathcal{L}_1) \otimes H^0(X, \mathcal{L}_2) \to H^0(X, \mathcal{L}_1 \otimes \mathcal{L}_2)$. Let E_1E_2 denote the span of all the products $f_1 f_2 \in H^0(X, \mathcal{L}_1 \otimes \mathcal{L}_2)$ for $f_1 \in E_1$, $f_2 \in E_2$. We call $(E_1E_2, \mathcal{L}_1 \otimes \mathcal{L}_2)$ the *product of two linear systems* $(E_1, \mathcal{L}_1)$, $(E_2, \mathcal{L}_2)$. With this product the collection $\tilde{\boldsymbol{K}}(X)$ of all the linear systems on X is a commutative semigroup.

Again fix a nonzero section $t \in E$ and let $L = \{f_s \colon s \in E\}$ be the corresponding subspace of rational functions. Define the *completion of the linear system* E to be the subspace $\bar{E} = \{ft \colon f \in \bar{L}\}$, where $\bar{L}$ is the completion of the subspace L (as defined above). If X is normal, one verifies that $\bar{E}$ still consists of holomorphic sections, i.e., $\bar{E} \subset H^0(X, \mathcal{L})$. One also verifies that for any rational function h we have $\overline{hL} = h\bar{L}$, from which it follows that $\bar{E}$ is well-define, i.e., is independent of the choice of the section t.

A linear system is said to have *no base locus* if the intersection of the supports of the divisors $D + (f)$, $\forall f \in L$, is empty. In other words, if $E \subset \mathcal{L}$ is a subspace of holomorphic sections representing a linear system then E has no base locus if for any $x \in X$ there is $s \in E$ with $s(x) \neq 0$.

Definition 1.4 (Intersection index of linear systems)**.** Let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be line bundles on X with linear systems $E_i \subset H^0(X, \mathcal{L}_i)$ for $i = 1, \ldots, n$ with no base locus. The *intersection index* $[E_1, \ldots, E_n]$ is the number of points in $D_1 \cap \cdots \cap D_n$ where D_i is a generic divisor in the linear system E_i, i.e., $D_i = \mathrm{Div}(s_i)$, where $0 \neq s_i$ is a generic element of E_i. For each i, fix a section $t_i \in E_i$ and let $L_i \in \mathbb{C}(X)$ be the subspace associated to E_i and t_i. One sees that $[E_1, \dots, E_n]$ is in fact equal to the intersection index $[L_1, \dots, L_n]$ of subspaces of rational functions and hence is well-defined.

The intersection index of linear systems enjoys properties similar to the intersection index of subspaces:

(1) The intersection index is symmetric with respect to permuting the arguments.
(2) The intersection index is multi-linear with respect to the product of linear systems.
(3) The intersection index $[E_1, \dots, E_n]$ does not change if we replace any of the E_i with an analogous linear system $\tilde{E}_i$ (in particular with the completion $\bar{E}_i$).
(4) As for subspaces of rational functions, for a linear system E on X let us define the *Hilbert function* by $H_E(k) = \dim(\overline{E^k})$. Then the limit
$$a(E) = \lim_{k\to\infty} H_E(k)/k^n$$
exists, and the self-intersection index $[E, \dots, E]$ is equal to $n!\, a(E)$.

1.2. Mixed volume and mixed integral. A function $F\colon \mathcal{V} \to \mathbb{R}$ on a (possibly infinite dimensional) vector space $\mathcal{V}$ is called a homogeneous polynomial of degree k if its restriction to any finite dimensional subspace of $\mathcal{V}$ is a homogeneous polynomial of degree k. (For any k, the constant zero function is a homogeneous polynomial of degree k.)

Definition 1.5. To a symmetric multi-linear function $B(v_1, \dots, v_k)$, $v_i \in \mathcal{V}$ one corresponds a homogeneous polynomial P of degree k on $\mathcal{V}$ defined by $P(v) = B(v, \dots, v)$. We say that the symmetric form B is a *polarization of the homogeneous polynomial* P.

If F is a homogeneous polynomial of degree k, then its derivative $F'_v(x)$ in the direction of a vector v is linear in v and homogeneous of degree $k-1$ in x. Let $v_1, \dots, v_k$ be a k-tuple of vectors. For each x, the k-th derivative $F^{(k)}_{v_1,\dots,v_k}(x)$ is a symmetric multi-linear function in the v_i. One easily verifies the following:

Proposition 1.6. *Any homogeneous polynomial of degree k has a unique* polarization B *defined by the formula:*
$$B(v_1, \dots, v_k) = (1/k!)F^{(k)}_{v_1,\dots,v_k}. \tag{2}$$

A compact convex subset of $\mathbb{R}^n$ is called a *convex body*. Consider the collection of convex bodies in $\mathbb{R}^n$. There are two operations of Minkowski sum and multiplication by a non-negative scalar on convex bodies. The collection of convex bodies with Minkowski sum is a semigroup with cancelation. The multiplication by a non-negative scalar is associative and distributive with respect to the Minkowski sum. These properties allow us to extend the collection of convex bodies to the (infinite dimensional) vector space $\mathcal{V}$ of *virtual convex bodies* consisting of formal differences of convex bodies (see [4]).

Let $d\mu = dx_1 \dots dx_n$ be the standard Euclidean measure in $\mathbb{R}^n$. For each convex body $\Delta \subset \mathbb{R}^n$ let $\mathrm{Vol}(\Delta) = \int_\Delta d\mu$ be its volume. The following statement is well-known:

Proposition 1.7. *The function* Vol *has a unique extension to the vector space $\mathcal{V}$ of virtual convex bodies as a homogeneous polynomial of degree n.*

Definition 1.8. The *mixed volume* $V(\Delta_1, \dots, \Delta_n)$ of the convex bodies Δ_i is the value of the polarization of the volume polynomial Vol at $(\Delta_1, \dots, \Delta_n)$.

Fix a homogeneous polynomial F of degree p in $\mathbb{R}^n$. Let $IF(\Delta) = \int_\Delta F\, d\mu$ denote the integral of F on Δ. One has the following (see for example [14]):

Proposition 1.9. *The function IF has a unique extension to the vector space $\mathcal{V}$ of virtual convex bodies as a homogeneous polynomial of degree $n + p$.*

Definition 1.10. The mixed integral $IF(\Delta_1, \dots, \Delta_{n+p})$ of a homogeneous polynomial F over the bodies $\Delta_1, \dots, \Delta_{n+p}$ is the value of the polarization of the polynomial IF at the bodies $\Delta_1, \dots, \Delta_{n+p}$.

From definition, the mixed integral of the constant polynomial $F \equiv 1$ is the mixed volume.

More generally we can consider the mixed volume and mixed integral for convex bodies in $\mathbb{R}^n$ which are parallel to a fixed subspace of $\mathbb{R}^n$. Fix a subspace $\Pi \subset \mathbb{R}^n$ with $\dim(\Pi) = m$. Consider the collection of convex bodies which are parallel to Π, i.e., lie in a translate $a + \Pi$ of Π for some $a \in \mathbb{R}^n$. This collection is closed under addition and multiplication by nonnegative scalars. Let $\mathcal{V}(\Pi)$ denote the subspace of all virtual convex bodies $\mathcal{V}$ spanned by the convex bodies parallel to Π. Fix a Lebesgue measure on Π and equip each translate of Π with a Lebesgue measure by shifting the measure on Π. We denote all these measures by $d\gamma$. Let $\Delta \subset a + \Pi$ be a convex body parallel to Π. The map

$$\Delta \mapsto \mathrm{Vol}_\Pi(\Delta)$$

is a homogeneous polynomial of degree m on the vector space $\mathcal{V}(\Pi)$, where Vol_Π is the volume with respect to the Lebesgue measure $d\gamma$. We will denote the polarization of Vol_Π on $\mathcal{V}(\Pi)$ by V_Π and call it the *mixed volume of convex bodies parallel to* Π.

Similarly, let F be a homogeneous polynomial on $\mathbb{R}^n$ of degree d. Then the map

$$\Delta \mapsto \int_\Delta F\, d\gamma$$

is a homogeneous polynomial on $\mathcal{V}(\Pi)$. We will denote the polarization of this by IF_Π. It is a $(m + d)$-linear function on $\mathcal{V}(\Pi)$.

1.3. Semigroup of finite sets with respect to addition. There is an addition operation on the collection of subsets of $\mathbb{R}^n$. The sum of two sets A and B is the set $A + B = \{a + b\colon a \in A,\ b \in B\}$. One verifies that the sum of two convex bodies (respectively convex integral polytopes) is again a convex body (respectively a convex integral polytope). This is the well-known Minkowski sum of convex bodies. Consider the following:

- $\mathcal{S}$, the semigroup of all finite subsets of $\mathbb{Z}^n$ with the addition of subsets.
- $\mathcal{P}$, the semigroup of all convex integral polytopes with the Minkowski sum.

Proposition 1.11. *The semigroup $\mathcal{P}$ has cancelation property.*

Proposition 1.11 follows from the more general fact that the semigroup of convex bodies with respect to the Minkowski sum has cancelation property. The next statement is easy to verify:

Proposition 1.12. *The map which associates to a finite nonempty set $A \subset \mathbb{Z}^n$ its convex hull $\Delta(A)$, is a homomorphism of semigroups from $\mathcal{S}$ to $\mathcal{P}$.*

For an integral convex polytope $\Delta \in \mathcal{P}$ let $\Delta_{\mathbb{Z}} \in \mathcal{S}$ denote the finite set of integral points in Δ, i.e., $\Delta_{\mathbb{Z}} = \Delta \cap \mathbb{Z}^n$. It is not hard to verify the following (see [11]):

Proposition 1.13. *For any nonempty subset $A \subset \mathbb{Z}^n$ we have:*

$$A + n\Delta(A)_{\mathbb{Z}} = (n+1)\Delta(A)_{\mathbb{Z}} = \Delta(A)_{\mathbb{Z}} + n\Delta(A)_{\mathbb{Z}}.$$

We then have the following description for the Grothendieck semigroup of $\mathcal{S}$.

Theorem 1.14. *The Grothendieck semigroup of $\mathcal{S}$ is isomorphic to $\mathcal{P}$. The homomorphism $\rho\colon \mathcal{S} \to \mathcal{P}$ is given by $\rho(A) = \Delta(A)$.*

Proof. From Propositions 1.11 and 1.12 it follows that if $A \sim B$ then $\Delta(A) = \Delta(B)$. Conversely, from Proposition 1.13 we know that A and $\Delta(A)_{\mathbb{Z}}$ are analogous. By definition if $\Delta(A) = \Delta(B)$ then $\Delta(A)_{\mathbb{Z}} = \Delta(B)_{\mathbb{Z}}$ and hence A and B are analogous. □

2. Horospherical homogeneous spaces

2.1. Horospherical subgroups. Recall that G denotes a connected reductive algebraic group.

Definition 2.1 (Horospherical subgroup)**.** A subgroup $H \subset G$ is called *horospherical* if it contains a maximal unipotent subgroup. The corresponding homogeneous space G/H is called a *horospherical homogeneous space*.

The horospherical spaces (respectively their equivariant partial compactifications called S-varieties) have features similar to algebraic torus (respectively toric varieties).

The next theorem gives a description of the horospherical subgroups of G. Recall that a subgroup P of G is parabolic if it contains a Borel subgroup.

Theorem 2.2. *Let H be a horospherical subgroup of G. Then there exists a parabolic subgroup P of G such that $P' \subset H \subset P$, where P' denotes the commutator subgroup of P. Conversely, any closed subgroup H with $P' \subset H \subset P$ is horospherical.*

Proof. Let H be a horospherical subgroup containing a maximal unipotent subgroup U. By Chevalley's theorem we can find a finite dimensional G-module V and a vector $0 \neq v \in V$ such that H is the stabilizer of the point $[v]$ in the projective space $\mathbb{P}(V)$. Since $U \subset H$ and U has no characters we see that v is fixed by U

and hence should be a sum of highest weight vectors. Let us write $v = \sum_{i=1}^{s} v_i$, where each v_i is a highest weight vector of some weight λ_i. The Borel B stabilizes the point $x = ([v_1], \ldots, [v_s]) \in \prod_{i=1}^{s} \mathbb{P}(V_{\lambda_i})$ and hence the stabilizer subgroup P of x is a parabolic subgroup. Now since H also stabilizes x we have $H \subset P$ as required. Finally, the characters λ_i restrict trivially to P' and thus P' fixes the point $[v] \in \mathbb{P}(\bigoplus_{i=1}^{s} V_{\lambda_i})$, which proves that $P' \subset H$. To prove the converse statement we need to show that $U \subset P'$. But U is the commutator of B and $B \subset P$. This finishes the proof. □

2.2. Semigroup of invariant subspaces of $\mathbb{C}[G/P']$. Fix a Borel subgroup B and let U be its maximal unipotent subgroup. One knows that there is a one-to-one correspondence between the parabolic subgroups containing B and the faces of the positive Weyl chamber $\Lambda^+_{\mathbb{R}}$. Let σ be a face of the positive Weyl chamber $\Lambda^+_{\mathbb{R}}$. Let $\sigma_{\mathbb{R}}$ denote the linear span of the cone σ. Also let $\Lambda_\sigma = \Lambda \cap \sigma_{\mathbb{R}}$ denote the lattice of weights lying on $\sigma_{\mathbb{R}}$ and let $\Lambda^+_\sigma = \Lambda^+ \cap \sigma$ be the semigroup of dominant weights lying on the face σ. Let P be the parabolic subgroup containing B which corresponds to σ and P' its commutator subgroup. The inclusion $i\colon B \hookrightarrow P$ induces a map $i^*\colon \mathfrak{X}(P) \to \mathfrak{X}(B) = \Lambda$. The following is well-known:

Proposition 2.3. *The map i^* is an inclusion, i.e., each character of P is determined by its restriction to B (equivalently T). Moreover, the image of i^* coincides with the lattice Λ_σ, i.e., the characters which lie on the linear span of the face σ. In particular, the rank of the lattice $\mathfrak{X}(P)$ is equal to the dimension of the face σ.*

We will identify the character group $\mathfrak{X}(P)$ with Λ_σ.

Consider the quotient group $S = P/P'$. By definition of P', S is an abelian algebraic group. The natural projection map $\pi\colon P \to S$ gives a map $\pi^*\colon \mathfrak{X}(S) \to \mathfrak{X}(P)$.

Proposition 2.4. *The group S is a torus of dimension equal to the dimension of the face σ. Moreover, the map π^* gives an isomorphism between the character lattice of S and the lattice Λ_σ*

We will also identify $\mathfrak{X}(S)$ with Λ_σ.

Now consider the homogeneous space $X = G/P'$. As P' is a normal subgroup of P, the group P and hence $S = P/P'$ act on X from right. Also G acts on X from left and the two actions commute.

The following theorem [16] is well-known and plays an important role for us.

Theorem 2.5. (1) *The variety X is a quasi-affine variety.*

(2) *The ring of regular functions $\mathbb{C}[X]$ decomposes as:*

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda^+_\sigma} W_\lambda,$$

where W_λ denotes the λ-eigenspace for the action of S. Moreover, as a G-module W_λ is isomorphic to the dual representation V^*_λ.

(3) *For any two dominant weights $\lambda, \mu \in \Lambda^+_\sigma$ we have*

$$W_\lambda W_\mu = W_{\lambda+\mu}.$$

Definition 2.6. (1) Let $0 \neq f \in \mathbb{C}[X]$. Then we can write $f = \sum_{\lambda \in \Lambda_\sigma^+} f_\lambda$, where $f_\lambda \in W_\lambda \cong V_\lambda^*$. The *support of* f is the collection $\mathrm{supp}(f)$ of all the dominant weights λ for which $f_\lambda \neq 0$. We define the support of the 0 function to be the empty set.

(2) Let $L \subset \mathbb{C}[X]$ be a G-invariant subspace of regular functions on X (which is also automatically invariant under the right S-action). The *support of* L is the collection $\mathrm{supp}(L)$ of dominant weights such that

$$L = \bigoplus_{\lambda \in \mathrm{supp}(L)} W_\lambda.$$

In other words, $\mathrm{supp}(L)$ is the union of all the $\mathrm{supp}(f)$ for $f \in L$.

(3) Let $A \subset \Lambda_\sigma^+$ be a finite set. Put

$$L_A = \bigoplus_{\lambda \in A} W_\lambda.$$

In other words, L_A is the collection of all the $f \in \mathbb{C}[X]$ with $\mathrm{supp}(f) \subset A$. By Theorem 2.5, L_A is a finite dimensional $(G \times S)$-invariant subspace of $\mathbb{C}[X]$.

Definition 2.7 (Moment polytope of a subspace)**.** For a G-invariant subspace $L \subset \mathbb{C}[X]$ we denote the convex hull of $\mathrm{supp}(L)$ by $\Delta(L)$ and call it the *moment polytope of* L.

Definition 2.8. We denote the collection of all the finite dimensional subspaces of $\mathbb{C}[X]$ which are invariant under the left G action by $\boldsymbol{K}_G(X)$.

The set $\boldsymbol{K}_G(X)$ is a semigroup under the product of subspaces. Moreover, if $L \in \boldsymbol{K}_G(X)$ is a G-invariant subspace then its integral closure $\overline{L}$ is also G-invariant, i.e., $\overline{L} \in \boldsymbol{K}_G(X)$.

The next proposition follows immediately from Theorem 2.5.

Proposition 2.9. (1) *Let $L_1, L_2 \in \boldsymbol{K}_G(X)$ be two G-invariant subspaces. Then*

$$\mathrm{supp}(L_1 L_2) = \mathrm{supp}(L_1) + \mathrm{supp}(L_2),$$

and hence

$$\Delta(L_1 L_2) = \Delta(L_1) + \Delta(L_2).$$

(2) *Let $A_1, A_2 \subset \Lambda_\sigma^+$ be finite subsets. Then*

$$L_{A_1 + A_2} = L_{A_1} L_{A_2}.$$

Let $\mathcal{S}(\Lambda_\sigma^+)$ denote the semigroup of all finite subsets of Λ_σ^+ together with the operation of addition of subsets. Also let $\mathcal{P}(\Lambda_\sigma^+)$ be the semigroup of all convex polytopes in σ with vertices in Λ_σ^+ together with the Minkowski sum of convex sets. By Theorem 1.14, the map $A \mapsto \Delta(A)$, the convex hull of A, gives an isomorphism between the Grothendieck semigroup of $\mathcal{S}(\Lambda_\sigma^+)$ and the semigroup $\mathcal{P}(\Lambda_\sigma^+)$.

The following theorem is a corollary of Proposition 2.9

Theorem 2.10 (Semigroup of invariant subspaces)**.** (1) *The map $L \mapsto \mathrm{supp}(L)$ gives an isomorphism of the semigroup $\boldsymbol{K}_G(X)$ of invariant subspaces of regular functions and the semigroup $\mathcal{S}(\Lambda_\sigma^+)$ of finite subsets of Λ_σ^+.*

(2) *The map* $L \mapsto \Delta(L)$ *gives an isomorphism of the Grothendieck semigroup of* $\boldsymbol{K}_G(X)$ *and the semigroup* $\mathcal{P}(\Lambda_\sigma^+)$ *of convex lattice polytopes in* σ.

(3) *If* $L \in \boldsymbol{K}_G(X)$, *the completion* $\overline{L}$ *is given by*

$$\overline{L} = \bigoplus_{\lambda \in \Delta(L) \cap \Lambda_\sigma^+} W_\lambda.$$

Thus under the isomorphism in the part (1), $\overline{L}$ *corresponds to the finite set of all the dominant weights in the moment polytope* $\Delta(L)$.

According to the *Weyl dimension formula*, the dimension of an irreducible representation V_λ is equal to $F(\lambda)$, where F is a polynomial on $\mathbb{R}^r$ of degree $(d-r)/2$ defined explicitly in terms of data associated to the Weyl group W (recall that $d = \dim(G)$ and $r = \dim(T)$). We call F the *Weyl polynomial of* W. Let F_σ denote the restriction of F to the linear span $\sigma_\mathbb{R}$ of the face σ, and let ϕ_σ be the homogeneous component of F_σ of highest degree.

Using Theorem 2.10 we can now obtain a formula for the intersection index of invariant subspaces. Let $L \in \boldsymbol{K}_G(X)$ be an invariant subspace. By Theorem 1.3 (Hilbert's theorem) we have

$$[L, \ldots, L] = p! \lim_{k\to\infty} \frac{\dim(\overline{L^k})}{k^p}.$$

On the other hand, if Δ is the moment polytope of L, Theorem 2.10(3) implies that

$$\lim_{k\to\infty} \frac{\dim(\overline{L^k})}{k^p} = \lim_{k\to\infty} \frac{\sum_{\lambda\in k\Delta(L)\cap\Lambda_\sigma^+} F(\lambda)}{k^p} = \int_\Delta \phi_\sigma \, d\mu,$$

where $d\mu$ is the Lebesgue measure on $\sigma_\mathbb{R}$ normalized with respect to the lattice Λ_σ. Thus $[L, \ldots, L] = p! \int_\Delta \phi_\sigma \, d\mu$. Finally, from the multi-linearity of the intersection index (the equation (1) in Section 1.1) and the additivity of the moment polytope (Proposition 2.9), we obtain the following formula:

Corollary 2.11 (Intersection index of invariant subspaces)**.** *Let* $L_1, \ldots, L_p \in \boldsymbol{K}_G(X)$ *be* G*-invariant subspaces,* $p = \dim(X)$. *For each* i, *let* $\Delta_i = \Delta(L_i)$ *be the moment polytope of the subspace* L_i. *We have*

$$[L_1, \ldots, L_p] = p!\, I\phi_\sigma(\Delta_1, \ldots, \Delta_p),$$

where $I\phi_\sigma$ *is the mixed integral* (Section 1.2).

Remark 2.12. Note that each L_i is a subspace of regular functions and hence elements of the L_i do not have poles. Also as each L_i is G-invariant, the base locus of L_i (i.e., where all the elements of L_i vanish) is G-invariant. But G acts transitively on X and $L_i \neq \{0\}$, it follows that L_i has no base locus. That is, the intersection index $[L_1, \ldots, L_p]$ counts the number of solutions of a generic system in the whole X (see Definition 1.1).

2.3. Semigroup of G-invariant linear systems on G/H. Fix a Borel subgroup B with a maximal unipotent subgroup U. Let H be a subgroup of G containing U (i.e., H is a horospherical subgroup). In this section we consider the horospherical homogeneous space $Y = G/H$. We describe the semigroup of invariant linear systems on Y and its Grothendieck semigroup as well as the intersection index of such linear systems.

From Theorem 2.2 we know that there exists a parabolic subgroup P containing B such that $P' \subset H \subset P$. Let σ be the face of positive Weyl chamber corresponding to the parabolic subgroup P.

The inclusion $i\colon H \hookrightarrow P$ induces a restriction map $i^*\colon \mathfrak{X}(P) \to \mathfrak{X}(H)$. As in Proposition 2.3 identify $\mathfrak{X}(P)$ with the lattice Λ_σ and let $\Lambda(H) \subset \Lambda_\sigma$ be the kernel of the map i^*. Alternatively, H/P' is a subgroup of the torus $S = P/P'$ and $\Lambda(H)$ can be viewed as the kernel of the restriction map $\mathfrak{X}(S) \to \mathfrak{X}(H/P')$. Also let $\Lambda_{\mathbb{R}}(H) = \Lambda(H) \otimes \mathbb{R}$ denote the linear span of the lattice $\Lambda(H)$.

Let D be a divisor on X and $\{D + (f)\colon f \in L\}$ be a family of equivalent divisors (i.e a linear system) on X, where L is a finite dimensional subspace of rational functions. Let us assume that the family is invariant under the action of G, i.e., for each $g \in G$ and $f \in L$ we have $(g \cdot D) + (g \cdot f) = D + (h)$ for some $h \in L$. If we assume that the only regular nowhere zero functions on X are constants then the principal divisor (h) determines h up to a constant. One verifies that $g\colon f \mapsto h$ gives a projective representation of the group G in the projective space $\mathbb{P}(L)$. For simplicity let us assume that this lifts to a linear representation of G on L. Thinking of a linear system as a subspace of sections of the line bundle $\mathcal{L}$ (associated to the divisor D) we have the following definition. (Recall that a G-linearized line bundle $\mathcal{L}$ on Y is a line bundle $\mathcal{L}$ with an action of G on $\mathcal{L}$ extending its action on Y such that for any $x \in X$ the action of $g \in G$ maps the fibre $\mathcal{L}_x$ linearly to the fibre $\mathcal{L}_{g\cdot x}$.)

Definition 2.13 (G-invariant linear system)**.** We call a G-invariant finite dimensional subspace $E \subset H^0(Y, \mathcal{L})$, a *$G$-invariant linear system on Y*. We denote the collection of all such pairs $(E, \mathcal{L})$ (up to isomorphism) by $\tilde{\boldsymbol{K}}_G(Y)$.

The product of two G-invariant linear systems is again invariant and hence the set $\tilde{\boldsymbol{K}}_G(Y)$ is a semigroup with respect to the product of linear systems. Moreover, if $(E, \mathcal{L}) \in \tilde{\boldsymbol{K}}_G(Y)$ then the completion $\bar{E}$ is G-invariant, i.e., $(\bar{E}, \mathcal{L}) \in \tilde{\boldsymbol{K}}_G(Y)$.

Definition 2.14 (Support of an invariant linear system)**.** Let E be a G-invariant linear system on Y. The *support of E* is the set $\mathrm{supp}(E)$ of all dominant weights $\lambda \in \Lambda_\sigma^+$ for which V_λ^* appears in the decomposition of E into irreducible G-modules.

Definition 2.15 (Moment polytope of a linear system)**.** Let E be a G-invariant linear system. We call the convex hull of $\mathrm{supp}(E)$, the *moment polytope of E* and denote it by $\Delta(E)$.

Consider the natural projection $\pi\colon X = G/P' \to Y = G/H$. We would like to look at the pull-back $\pi^*(E)$ of a G-invariant linear system E on Y to X.

Without loss of generality we can assume that every hypersurface in G is given by an equation, that is, $\mathrm{Pic}(G) = \{0\}$. In fact, by a theorem of Popov (see [13]) for any connected linear algebraic group G there exists a central isogeny $\pi\colon \tilde{G} \to G$ such that $\mathrm{Pic}(\tilde{G}) = \{0\}$. (That is, π is an algebraic homomorphism such that $\ker(\pi)$ is finite and lies in the center of G.) Now if U is a maximal unipotent subgroup of G then $\pi^{-1}(U)$ is also a maximal unipotent subgroup of $\tilde{G}$. Thus replacing G

with $\tilde{G}$ we can assume that the Picard group of G is trivial. The next theorem is an immediate corollary of another result in [13]:

Theorem 2.16. *The character group $\mathfrak{X}(P')$ is trivial and hence any G-linearized line bundle on $X = G/P'$ is G-equivariantly isomorphic to the trivial bundle.*

Let $\mathcal{L}$ be a G-linearized line bundle on Y. Then by the above theorem the pull-back line bundle $\pi^*(\mathcal{L})$ is trivial. Thus we can identify the space of sections $H^0(X, \pi^*(\mathcal{L}))$ with the ring of regular functions $\mathbb{C}[X]$. Note that since $\pi\colon X \to Y$ is surjective, the map $\pi^*\colon H^0(Y, \mathcal{L}) \to H^0(X, \pi^*(\mathcal{L})) \cong \mathbb{C}[X]$ is one-to-one. The map π^* then identifies a G-invariant linear system E on Y with a G-invariant subspace $L(E) \subset \mathbb{C}[X]$. It is clear that $\operatorname{supp}(E) = \operatorname{supp}(L(E))$.

Theorem 2.17. *The map $E \mapsto L(E)$ gives a one-to-one correspondence between the collection of G-invariant linear systems on Y (up to isomorphism) and the collection of finite dimensional subspaces of $\mathbb{C}[X]$ which are invariant under the left action of G and lie in an eigenspace for the right action of H.*

Proof. Each character $\gamma \in \mathfrak{X}(H)$ gives a G-linearized line bundle $\mathcal{L}_\gamma$ on $Y = G/H$ defined by $\mathcal{L}_\gamma = (G \times \mathbb{C})/H$, where $h \in H$ acts on $(g, x) \in G \times \mathbb{C}$ by $h \cdot (g, x) = (gh^{-1}, \gamma(h)x)$. The fibration $\mathcal{L}_\gamma \to G/H$ is given by the projection on first factor. From definition, each holomorphic section of $\mathcal{L}_\gamma$ corresponds to a section of the trivial bundle over G which is invariant under the above action of H. It follows that the holomorphic sections of $\mathcal{L}_\gamma$ are in one-to-one correspondence with regular functions in $\mathbb{C}[G]$ which are γ-eigenfunctions for the right action of H. By a theorem of Popov [13] the correspondence $\gamma \mapsto \mathcal{L}_\gamma$ is an isomorphism of $\mathfrak{X}(H)$ and $\operatorname{Pic}_G(G/H)$, the group of G-linearized line bundles on G/H (with tensor product). Thus if $E \subset H^0(Y, \mathcal{L}_\gamma)$ is an invariant linear system, where $\mathcal{L}$ is a G-linearized line bundle, then for some character $\gamma \in \mathfrak{X}(H)$ we have $\mathcal{L} = \mathcal{L}_\gamma$ and E can be identified with a (left) G-invariant subspace of γ-eigenspace of $\mathbb{C}[G]$ for the right action of H. Also as P' has no characters, each γ-eigenfunction is P'-invariant and hence belongs to $\mathbb{C}[G/P']$. This proves the proposition. □

The next proposition describes the support of the subspace $L(E)$ associated to an invariant system E.

Proposition 2.18. (1) *Let $\gamma \in \mathfrak{X}(H)$ be a character of H. Let L be a (left) G-invariant subspace of $\mathbb{C}[G/P']$ consisting of γ-eigenfunctions of (right) action of H. Then $\operatorname{supp}(L)$ is contained in a coset of $\Lambda(H)$, i.e., for any $\lambda, \mu \in \operatorname{supp}(L)$ we have $\lambda - \mu \in \Lambda(H)$. In particular, the smallest affine space spanned by $\operatorname{supp}(L)$ is parallel to the linear space $\Lambda_{\mathbb{R}}(H)$.*

(2) *Conversely, let $A \subset \Lambda^+_\sigma$ be a finite subset which is contained in a coset of $\Lambda(H)$. Then the subspace $L_A = \bigoplus_{\lambda \in A} W_\lambda \subset \mathbb{C}[G/P']$ is contained in some eigenspace of (right) action of H.*

Proof. Let $\lambda, \mu \in \operatorname{supp}(L)$ with W_λ, W_μ the corresponding eigenspaces in $\mathbb{C}[G/P']$ for the right action of P. Then the functions in W_λ and W_μ are automatically eigenfunctions for the action of H with weights $i^*(\lambda)$ and $i^*(\mu)$ respectively, where $i^*\colon \mathfrak{X}(P) \to \mathfrak{X}(H)$ is the restriction of characters. On the other hand, every function in L is an eigenfunction for H with weight γ. This shows that $i^*(\lambda) = i^*(\mu) = \gamma$ and thus $i^*(\lambda - \mu) = 0$, i.e., $\lambda - \mu \in \Lambda(H)$. This proves (1). Now let $A \subset \Lambda_\sigma^+$ lay in a coset of $\Lambda(H)$. Then $i^*(A)$ consists of a single point $\gamma \in \mathfrak{X}(H)$. Then $L_A = \bigoplus_{\lambda \in A} W_\lambda \subset \mathbb{C}[G/P']$ consists of eigenfunctions of H with weight γ. This finishes the proof of (2). $\square$

From the previous proposition it follows that the moment polytope of E lies in an affine subspace parallel to $\Lambda_\mathbb{R}(H)$.

Consider the collection $\mathcal{S}(\Lambda(H))$ consisting of finite subsets A of Λ_σ such that for any $\lambda, \mu \in A$ we have $\lambda - \mu \in \Lambda(H)$ (i.e., A lies in a coset of $\Lambda(H)$). Clearly $\mathcal{S}(\Lambda(H))$ is a semigroup under the addition of subsets. Also consider the collection $\mathcal{P}(\Lambda(H))$ of all the convex lattice polytopes Δ in Λ_σ such that for any two vertices λ, μ of Δ we have $\lambda - \mu \in \Lambda(H)$. It is also clear that $\mathcal{P}(\Lambda(H))$ is a semigroup with respect to the Minkowski sum of convex bodies.

As in Theorem 1.14 we have:

Proposition 2.19. *The map $A \mapsto \Delta(A)$, the convex hull of A, gives an isomorphism between the Grothendieck semigroup of $\mathcal{S}(\Lambda(H))$ and the semigroup $\mathcal{P}(\Lambda(H))$.*

From Proposition 2.18 we get the following corollary:

Corollary 2.20. *The map $L \mapsto \mathrm{supp}(L)$ gives an isomorphism between the semigroup of finite dimensional subspaces of $\mathbb{C}[X]$ which are invariant under the left action of G and contained in some eigenspace for the right action of H, and the semigroup $\mathcal{S}(\Lambda(H))$ of finite subsets of Λ_σ which lie in a coset of $\Lambda(H)$.*

The next proposition follows immediately from Theorem 2.5 and Theorem 2.17.

Proposition 2.21. *Let $E_1, E_2 \in \tilde{\boldsymbol{K}}_G(Y)$ be two G-invariant linear systems. Then*

$$\mathrm{supp}(E_1E_2) = \mathrm{supp}(E_1) + \mathrm{supp}(E_2),$$

and hence

$$\Delta(E_1E_2) = \Delta(E_1) + \Delta(E_2).$$

Now, as in Theorem 2.10 we obtain the following description for the semigroup of invariant linear systems on Y as well as a description of the completion of a linear system.

Corollary 2.22 (Semigroup of G-invariant linear systems)**.** (1) *The map $E \mapsto \mathrm{supp}(E)$ gives an isomorphism between the semigroup $\tilde{\boldsymbol{K}}_G(Y)$ of G-invariant linear systems and the semigroup $\mathcal{S}(\Lambda(H))$ of finite subsets of Λ_σ which lie in a coset of $\Lambda(H)$.*

(2) *The map $E \mapsto \Delta(E)$ gives an isomorphism of the Grothendieck semigroup of $\tilde{\boldsymbol{K}}_G(Y)$ and the semigroup $\mathcal{P}(\Lambda(H))$ of convex lattice polytopes in $\sigma_\mathbb{R}$ whose set of vertices lie in a coset of $\Lambda(H)$.*

(3) *Let $\bar{E}$ be the completion of E. Then $\mathrm{supp}(\bar{E})$ is the intersection of the coset of $\Lambda(H)$ containing $\mathrm{supp}(E)$ and the moment polytope $\Delta(E)$.*

As before let ϕ_σ be the homogeneous component of highest degree of the Weyl polynomial F restricted to the linear span $\sigma_\mathbb{R}$ of the cone σ. Fix a Lebesgue measure on $\Lambda_\mathbb{R}(H)$ normalized with respect to $\Lambda(H)$, i.e., the smallest nonzero measure of a parallelepiped with vertices in $\Lambda(H)$ is equal to 1. We equip all the affine subspaces $a + \Lambda_{\mathbb{R}}(H)$, $a \in \sigma_{\mathbb{R}}$, with shifts of this Lebesgue measure.

Finally, similar to the proof of Corollary 2.11, using the properties of the intersection index of linear systems (paragraph after Definition 1.4), Corollary 2.22, and the additivity of the moment polytope (Proposition 2.21), we obtain the following formula:

Corollary 2.23 (Intersection index of invariant linear systems)**.** *Let $E_1, \ldots, E_m \in \boldsymbol{K}_G(Y)$ be G-invariant linear systems, where $m = \dim(Y)$. For each i, let $\Delta_i = \Delta(E_i)$ be a moment polytope of E_i. We have*

$$[E_1, \ldots, E_m] = m!\, I\phi_\sigma(\Delta_1, \ldots, \Delta_m),$$

where $I\phi_\sigma$ is the mixed integral of ϕ_σ for the polytopes which are parallel to the linear space $\Lambda_{\mathbb{R}}(H)$.

2.4. Intersection index as mixed volume. In this section we rewrite the formula for the intersection index as a mixed volume of certain polytopes (instead of mixed integral). To this end, we use the so-called *Gelfand–Cetlin polytopes*.

In their classical paper [5], Gelfand and Cetlin constructed a natural basis for any irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ and showed how to parameterize the elements of this basis with integral points in a certain convex polytope. These polytopes are called the *Gelfand–Cetlin polytopes*. Since then similar constructions have been done for other classical groups and analogous polytopes were defined (see [1]). We will also call them *Gelfand–Cetlin polytopes* or for short *G-C polytopes*. Consider the list of Lie algebras: the abelian algebra $\mathbb{C}^n$, $\mathfrak{sl}(n_1, \mathbb{C})$, $\mathfrak{so}(n_2, \mathbb{C})$ and $\mathfrak{sp}(2n_3, \mathbb{C})$, for any $n, n_1, n_2, n_3 \in \mathbb{N}$. We say that a connected reductive group is a *classical group*, if its Lie algebra is a direct sum of the algebras in this list. In this sense, the general linear group and the spinor group are classical groups.

Let G be a classical group. As usual let $d = \dim(G)$. We have:

Theorem 2.24 (G-C polytopes)**.** *For any classical group G and for any $\lambda \in \Lambda^+_{\mathbb{R}}$ one can explicitly construct a polytope $\Delta_{\mathrm{GC}}(\lambda) \subset \mathbb{R}^{(d-r)/2}$, called the Gelfand–Cetlin polytope of λ, with the following properties*:

(1) *If λ is integral then the dimension of V_λ is equal to the number of integral points in $\Delta_{\mathrm{GC}}(\lambda)$.*
(2) *The map $\lambda \mapsto \Delta_{\mathrm{GC}}(\lambda)$ is linear, i.e., for any two $\lambda, \gamma \in \Lambda^+_{\mathbb{R}}$ and $c_1, c_2 \geqslant 0$ we have $= \Delta_{\mathrm{GC}}(c_1\lambda + c_2\gamma) = c_1\Delta_{\mathrm{GC}}(\lambda) + c_2\Delta_{\mathrm{GC}}(\gamma)$.*

The part (2) in the above theorem is an immediate corollary of the defining inequalities of the G-C polytopes for the classical groups.

Definition 2.25 (Newton polytope)**.** Let A be a finite nonempty set of dominant weights in Λ^+_σ. Define the polytope $\tilde{\Delta}(A) \subset \sigma \times \mathbb{R}^{(d-r)/2}$ by:

$$\tilde{\Delta}(A) = \bigcup_{\lambda \in \Delta(A)} \{(\lambda, x) \colon x \in \Delta_{\mathrm{GC}}(\lambda)\}.$$

In other words, the projection on the first factor maps $\tilde{\Delta}(A)$ to the moment polytope $\Delta(A)$ and the fibre over each λ is the G-C polytope $\Delta_{\mathrm{GC}}(\lambda)$. For a G-invariant subspace $L \subset \mathbb{C}[X]$ we define its *Newton polytope* $\tilde{\Delta}(L)$ to be the polytope $\tilde{\Delta}(\mathrm{supp}(L))$.

Similarly we define a *Newton polytope* $\tilde{\Delta}(E)$ of a G-invariant linear system E to be the Newton polytope of $\mathrm{supp}(E)$.

From Corollaries 2.11 and 2.23 we obtain the following:

Corollary 2.26 (Intersection index of subspaces as mixed volume)**.** *Suppose that $L_1, \dots, L_p \in \boldsymbol{K}_G(X)$ are G-invariant subspaces, where $p = \dim(X)$. For each i, let $\tilde{\Delta}_i = \tilde{\Delta}(L_i)$ be the Newton polytope of the subspace L_i. Then we have*

$$[L_1, \dots, L_p] = p!\, V(\tilde{\Delta}_1, \dots, \tilde{\Delta}_p),$$

where V denotes the mixed volume of convex bodies in the cone $\sigma \times \mathbb{R}^{(d-r)/2}$.

Corollary 2.27 (Intersection index of linear systems as mixed volume)**.** *Suppose that $E_1, \dots, E_m \in \tilde{\boldsymbol{K}}_G(Y)$ are G-invariant linear systems, where $m = \dim(Y)$. For each i, let $\tilde{\Delta}_i = \tilde{\Delta}(E_i)$ be a Newton polytope of E_i. Then we have*

$$[E_1, \dots, E_m] = m!\, V_H(\tilde{\Delta}_1, \dots, \tilde{\Delta}_m),$$

where V_H denotes the mixed volume of convex bodies in $\sigma \times \mathbb{R}^{(d-r)/2}$ and parallel to $\Lambda_{\mathbb{R}}(H) \times \mathbb{R}^{(d-r)/2}$.

2.5. Case of $\mathbf{GL}(\boldsymbol{n}, \mathbb{C})$. Let $G = \mathrm{GL}(n, \mathbb{C})$ and let B be the Borel subgroup of upper triangular matrices. Then the subgroup T of diagonal matrices is a maximal torus, and the subgroup of upper-triangular matrices with 1's on the diagonal is the maximal unipotent subgroup contained in B. We identify the weight lattice Λ with $\mathbb{Z}^n$ and its linear span $\Lambda_{\mathbb{R}}$ with $\mathbb{R}^n$. The Weyl group of G is identified with the symmetric group S_n acting on $\mathbb{R}^n$ by permuting the coordinates. The positive Weyl chamber for the choice of B is $\Lambda^+_{\mathbb{R}} = \{\lambda = (\lambda_1, \dots, \lambda_n) \colon \lambda_1 \leqslant \cdots \leqslant \lambda_n\}$.

There is a one-to-one correspondence between the subsets $I = \{k_1 < \cdots < k_s\}$ of $\{1, \dots, n-1\}$ and the faces

$$\sigma_I = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \Lambda^+_{\mathbb{R}} \colon \lambda_{k_i+1} = \lambda_{k_i+2} = \cdots = \lambda_{k_{i+1}},\ \forall\, i = 0, \dots, s\}$$

of the positive Weyl chamber. Here by convention $k_0 = 0$ and $k_{s+1} = n$. Also each subset I then corresponds to a parabolic subgroup P_I consisting of the block upper-triangular matrices with blocks of fixed sizes $k_1, k_2 - k_1, \dots, k_s - k_{s-1}, n - k_s$. One verifies that the commutator subgroup P_I' consists of the block upper-triangular matrices, where determinant of each block is equal to 1.

Moreover, the torus $S = P_I/P_I'$ can be identified with $(\mathbb{C}^*)^{s+1}$ and the natural map $P_I \to P_I/P_I'$ is given by $x \mapsto (\det(B_1), \dots, \det(B_{s+1}))$, where $x \in P$ and $B_1, \dots, B_{s+1}$, are the blocks of sizes $k_1, k_2 - k_1, \dots, k_s - k_{s-1}, n - k_s$ respectively sitting on the diagonal of x.

Let us see that G/P_I' is quasi-affine by giving an embedding of this variety in some affine space. Let $g \in G$ be an invertible matrix with columns $C_1, \dots, C_n$. Consider the map $\Psi \colon G \to \bigoplus_{i=1}^{s+1} \bigwedge^{k_i} \mathbb{C}^n$ given by

$$g \mapsto \sum_{i=1}^{s+1} (C_1 \wedge \cdots \wedge C_{k_i}).$$

One verifies that Ψ induces an embedding of G/P_I' into the affine space $\bigoplus_{i=1}^{s+1} \bigwedge^{k_i} \mathbb{C}^n$. This map is closely related to the generalized Plücker embedding.

Since G is a Zariski open subset of $\mathbb{C}^{(n^2)}$, it is clear that $\mathrm{Pic}(G) = \{0\}$.

In the next example we consider a special case of a horospherical homogeneous space which is related to the classical Bezout theorem.

Example 2.28. Let $G = \mathrm{GL}(n, \mathbb{C})$ act on $\mathbb{C}^n$ in the natural way. Then $\mathbb{C}^n \setminus \{0\}$ is an orbit O. Let H be the G-stabilizer of e_1, the first standard basis vector. It contains the subgroup of upper triangular matrices with 1's on the diagonal, i.e., H is a horospherical subgroup and O is a horospherical homogeneous space G/H. If $n > 1$, the space of regular functions $\mathbb{C}[O]$ is isomorphic to the polynomial algebra on $\mathbb{C}^n$ and as a G-module it decomposes into

$$\mathbb{C}[O] = \bigoplus_{k=1}^{\infty} V_k,$$

where V_k is the space of homogeneous polynomials of degree k on $\mathbb{C}^n$. It is the dual of the k-th symmetric power of the tautological representation of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n$. For each $k \geqslant 0$, V_k is an irreducible representation with highest weight $(0 = \cdots = 0 \leqslant k)$ (under the identification of the dominant weights of $\mathrm{GL}(n, \mathbb{C})$ with non-decreasing sequences of integers $\lambda = (\lambda_1 \leqslant \cdots \leqslant \lambda_n)$). Let $F(k) = \dim(V_k) =$ number of monomials in n variables and of total degree k. One knows that $F(k) = \binom{k+n-1}{n-1}$. Then $\phi(k) = k^{n-1}/(n-1)!$ is the homogenous component of F of highest degree. For each finite set $A = \{a_1, \ldots, a_s\} \subset \mathbb{Z}_{\geqslant 0}$ let L_A be the space of polynomials on $\mathbb{C}^n$ whose homogeneous components have degrees $a_1, \ldots, a_s$, i.e., $\mathrm{supp}(L_A) = A$. Let $A_1, \ldots, A_n \subset \mathbb{Z}_{\geqslant 0}$ be finite subsets and for each i, let $f_i \in L_{A_i}$ be a generic polynomial. Then by Corollary 2.11 the number of solutions of the system $f_1(x) = \cdots = f_n(x) = 0$ in $\mathbb{C}^n \setminus \{0\}$ is equal to $n!\, I\phi(I_1, \ldots, I_n)$, where for each i, $I_i = \Delta(A_i) = [a_i, b_i]$ is the interval which is the convex hull of the finite set A_i. Using the equation (2) in Section 1.2, we can compute the mixed integral:

$$[L_{A_1}, \ldots, L_{A_n}] = n!\, I\phi(I_1, \ldots, I_n) = \prod_{i=1}^{n} b_i - \prod_{i=1}^{n} a_i.$$

This can be thought of as an affine version of the classical Bezout theorem generalized to arbitrary dimensions. For each $k \geqslant 0$ let Δ_k be the G-C polytope associated to the dominant weight $(k \geqslant 0 = \cdots = 0)$. From the defining equations of G-C polytopes we see that $\Delta_{\mathrm{GC}}(k) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{n-1} : k \geqslant x_{n-1} \geqslant \cdots \geqslant x_1 \geqslant 0\}$. Now for a finite subset $A \subset \mathbb{Z}_{\geqslant 0}$, let $I = \Delta(A) = [a, b]$. Then the Newton polytope $\tilde{\Delta}(A) \subset \mathbb{R}^n$ is defined by

$$\tilde{\Delta}(A) = \{(k, x_1, \ldots, x_{n-1}) : k \geqslant x_{n-1} \geqslant \cdots \geqslant x_1 \geqslant 0,\ a \geqslant k \geqslant b\}.$$

Let $\tilde{\Delta}_i$ denote the Newton polytope of the finite subset A_i. Then by Corollary 2.26, $[L_{A_1}, \ldots, L_{A_n}]$ is also given by:

$$n!\, V(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_n).$$

Here V denotes the mixed volume of convex bodies in $\mathbb{R}^n$. One can verify that both of the above formulae agree with the answer obtained by using the Bernstein–Kushnirenko theorem. Note that in the above we count the solutions in $\mathbb{C}^n \setminus \{0\}$ while in the Bernstein–Kushnirenko we count the solutions in $(\mathbb{C}^*)^n$.

The last example concerns the degree of equivariant line bundles on partial flag varieties.

Example 2.29. Let $I \subset \{1, \ldots, n-1\}$ and let $H = P_I$ be the corresponding parabolic subgroup. Clearly H is a horospherical subgroup. Put $\dim(G/P_I) = m$. For a dominant weight $\lambda \in \Lambda^+_{\sigma_I}$ let $\mathcal{L}_\lambda$ be the corresponding G-linearized line bundle on G/P_I and $E_\lambda = H^0(G/P_I, \mathcal{L}_\lambda) \cong V^*_\lambda$ the corresponding complete linear system. One sees that $\Delta(E_\lambda) = \lambda$ and $\tilde{\Delta}(E_\lambda) = \Delta_{\mathrm{GC}}(\lambda)$. Let $\lambda_1, \ldots, \lambda_m \in \Lambda^+_{\sigma_I}$ be dominant weights with the corresponding linear systems $E_1, \ldots, E_m$. Then the intersection index of these linear systems is given by

$$[E_1, \ldots, E_m] = m!\, V(\Delta_{\mathrm{GC}}(\lambda_1), \ldots, \Delta_{\mathrm{GC}}(\lambda_m)),$$

where V is the mixed volume of convex bodies in $\mathbb{R}^m$.

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