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Intersection Theory and Hilbert Function*

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Dedicated to the memory of Vladimir Igorevich Arnold

Abstract. Birationally invariant intersection theory is a far-reaching generalization and extension of the Bernstein–Kushnirenko theorem. This paper presents transparent proofs of Hilbert's theorem on the degree of a projective variety and other related statements playing an important role in this theory. The paper is completely self-contained; we recall all necessary definitions and statements.

Key words: degree of projective variety, Hilbert function, intersection theory, Bernstein–Kushnirenko theorem.

Introduction. Hilbert's theorem (see Section 7) relates the degree of a projective variety to the asymptotic behavior of its Hilbert function. This paper provides elementary proofs of this theorem over the field of complex numbers and of other related statements, some of which are not obvious from the algebro-geometric point of view (Theorem 10).

Why should one reprove a classical theorem appearing in many textbooks on algebraic geometry? This fundamental theorem plays a key role in recently discovered birationally invariant intersection theory, both in its complex-analytic version and in its algebraic counterpart, which is valid over an arbitrary algebraically closed field. The algebraic version relies on the standard form of Hilbert's theorem. It is natural however to give an analytic proof for the analytic version. This proof provides a new perspective on the subject and suggests new algebraic results.

0.1. Birationally invariant intersection theory. The famous Bernstein–Kushnirenko theorem on the number of solutions in $(\mathbb{C}^*)^n$ of a generic system of n equations with fixed Newton polytope was found in 1975 (for a proof based on Hilbert's theorem, see [1] and [2]). Its discovery has been made possible by rich empirical evidence gathered by Vladimir Igorevich Arnold in his investigations of critical points of functions. The theorem was the beginning of the theory of Newton polytopes, which connected convex geometry with algebraic geometry and singularity theory in the framework of the geometry of toric varieties (see [3] and[4]). This theory has been extensively developed (in particular, in Arnold's seminar); it was the subject of hundreds of publications.

The Bernstein-Kushnirenko theorem does not fit into the framework of classical intersection theory: it deals with the number of solutions of a system of equations on the incomplete variety $(\mathbb{C}^*)^n$ whose left-hand sides are fairly generic elements of special finite-dimensional spaces. The answer is given in terms of Newton polytopes, whose very definition uses the specific character of these spaces and of the variety $(\mathbb{C}^*)^n$.

Instead of $(\mathbb{C}^*)^n$ birationally-invariant intersection theory considers an arbitrary (not necessarily complete) irreducible n-dimensional algebraic variety X and systems of equations on it whose left-hand sides are generic elements of arbitrary finite-dimensional spaces of rational functions on X. These finite-dimensional spaces form a multiplicative semigroup $\mathscr{K}(X)$. The intersection index of n spaces from $\mathscr{K}(X)$ is the (properly counted) number of solutions of a system of equations whose left-hand sides are generic elements of these n spaces. The intersection index automatically carries over to the Grothendieck group of the semigroup $\mathscr{K}(X)$. To each subspace we assign its *Newton–Okounkov body* $\Delta(L) \subset \mathbb{R}^n$ so that $\Delta(L_1) + \Delta(L_2) \subset \Delta(L_1L_2)$ and the self-intersection index of the subspace L is equal to the volume of the body $\Delta(L)$ multiplied by $n!$, like in Kushnirenko's

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theorem (the construction of the body $\Delta(L)$ is nonunique and contains arbitrary functional parameters). The theory relates algebra and geometry outside the framework of toric geometry (see [5] and [6]).

This relationship is useful in many directions. For algebraic geometry it provides elementary proofs of intersection-theoretic analogues of the geometric Alexandrov–Fenchel inequalities (see [5] and [8]) and far-reaching generalizations of the Fujita approximation theorem (see [6] and [8]). In invariant theory it gives analogues of the Bernstein–Kushnirenko theorem for horospherical varieties [9] and some other manifolds with an action of a reductive group (see [10] and [11]) and makes it possible to compute the Grothendieck group of the semigroup of representations (considered up to spectral equivalence) of a reductive group [10]. In abstract algebra it allows one to introduce a broad class of graded algebras whose Hilbert functions are not necessarily polynomial at large values of the argument but have polynomial asymptotics characterized by constants satisfying an analogue of the Brunn-Minkowski inequalities [8]. In geometry it provides a transparent proof of the Alexandrov–Fenchel inequality and its numerous consequences (see [5] and [8]). This relationship is based on the geometric theory of semigroups of integral points [8].

The part of the complex-analytic version of this theory containing its global version and the version related to an action of a reductive group has been published (see the list of references). K. Kaveh and I are completing a local version, which provides new inequalities (similar to the Alexandrov–Fenchel inequalities) for the multiplicities of primary ideals in a local ring. Considerations of this paper are applied there to provide a topological proof of the Samuel formula for the multiplicity of a root of a system of equations at a singular point of a variety.

We are also in the process of writing down a version of the theory valid over an arbitrary algebraically closed field (in which we use the standard form of Hilbert's theorem). We also obtain abstract analogues of all results, including analogues of our new inequalities for the multiplicities of primary ideals in abstract local rings.

0.2. Contents of the paper. Let X be an irreducible n-dimensional complex algebraic variety, and let $\mathscr{K}(X)$ be the multiplicative semigroup of nonzero finite-dimensional complex subspaces of the field $\mathbb{C}(X)$ of rational functions on X. The *Hilbert function* H_L and the *normalized Hilbert function* $\overline{H}_L$ of a subspace $L \in \mathscr{K}(X)$ are defined by $H_L(k) = \dim_{\mathbb{C}} L^k$ and $\overline{H}_L(k) = \dim_{\mathbb{C}} \overline{L^k}$, where $\overline{L^k}$ is the integral closure of the subspace L^k in the field $\mathbb{C}(X)$ (see Sections 2 and 6). The asymptotic behavior of the functions $H_L \leqslant \overline{H}_L$ is related to the self-intersection index d of the subspace L.

The situation is simplest when the subspace L separates the generic points of X. In this case there are upper and lower bounds for the functions $H_L \leqslant \overline{H}_L$ having the same asymptotic behavior as $k \to \infty$ and depending only on n and d (not on X and L). The proofs are based on properties of the intersection index on algebraic varieties and on simple arguments valid for arbitrary analytic varieties.

It follows from the bounds that

$$n! \lim_{k\to\infty} \frac{H_L(k)}{n^k} = d \quad \text{and} \quad n! \lim_{k\to\infty} \frac{\overline{H}_L(k)}{n^k} = d.$$

The former relation is equivalent to Hilbert's theorem for an irreducible projective variety over the field $\mathbb{C}$ and provides the simplest (among those known to me) proof of this theorem. For reducible projective varieties, there are no two-sided bounds for the Hilbert function that depend only on n and d and have the required asymptotic behavior.

If a subspace $L \in \mathscr{K}(X)$ does not separate the generic points of X, then the relation $n! \lim_{k\to\infty} H_L(k)/n^k = d$ does not hold. If $d > 0$, then, first, there is an upper bound for $\overline{H}_L$ depending only on d and n and having the required asymptotic behavior, and, secondly, $d = n! \lim \overline{H}_L(k)/n^k$ (there is however no lower bound of this kind for $\overline{H}_L$). In the general case, without the assumption $d > 0$, even the upper bound of this kind for $\overline{H}_L$ does not exist; however, we still have $d = n! \lim \overline{H}_L(k)/n^k$. It is this equality which relates the asymptotic behavior of the normalized Hilbert function $\overline{H}_L$ and the self-intersection index of the subspace L.

0.3. Structure of the paper. We recall the necessary definitions and statements of birationally invariant intersection theory in Sections 1 and 2. In Section 3 we introduce notation that is used in Sections 4 and 5 for estimating the dimensions of some spaces of analytic functions. In Section 6 we recall properties of integral closure and give a simple proof of the fact that $\dim_{\mathbb{C}} \overline{L} < \infty$ for $L \in \mathscr{K}(X)$. Section 7 deals with the case where L separates the generic points of X, and Section 8, with the case where $d > 0$. In Section 9 we prove the relation between the asymptotic behavior of the function $\overline{H}_L$ and the self-intersection index in the general case.

1. Intersection index on the semigroup $\mathscr{K}(X)$. Let X be an irreducible n-dimensional algebraic variety (in this paper by an algebraic variety we mean a quasi-projective algebraic variety) over the field $\mathbb{C}$. Let $\mathscr{K}(X)$ be the set of nonzero finite-dimensional subspaces over $\mathbb{C}$ of the field $\mathbb{C}(X)$ of rational functions on X. The set $\mathscr{K}(X)$ forms a *commutative semigroup* under the following multiplication operation: the *product of spaces* $L, M \in \mathscr{K}(X)$ is the space $LM \in \mathscr{K}(X)$ spanned by all the functions of the form fg with $f \in L$ and $g \in M$.

For every set of spaces $L_1, \dots, L_n \in \mathscr{K}(X)$, the *intersection index* $[L_1, \dots, L_n]$ is defined (see [7]): the number $[L_1, \dots, L_n]$ is equal to the number of solutions in X of a system of the form $f_1 = \dots = f_n = 0$, where $f_1 \in L_1, \dots, f_n \in L_n$ are a generic set of functions from the spaces $L_1, \dots, L_n$; the solutions at which all functions from some space L_i, $1 \leqslant i \leqslant n$, vanish, i.e., those solutions $a \in X$ for which $f(a) = 0$ for all $f \in L_i$, are not counted. The solutions at which at least one of the functions g from one of the spaces L_j has a pole are not counted either. Below we recall the main properties of the intersection index (see [7]).

(1) Intersection index is well defined. We say that a property holds for a *generic* element of a linear space M over $\mathbb{C}$ if there is a complex semialgebraic set $\Sigma \subset M$ such that this property holds for all elements of $M \setminus \Sigma$ and $\dim \Sigma < \dim M$. Let $L_1, \dots, L_n \in \mathscr{K}(X)$ be any set of subspaces, and let $O \subset X$ be any complex semialgebraic set such that

(a) O contains all singular points of the variety X;

(b) for $1 \leqslant i \leqslant n$, the inclusion $O_i \subset O$ holds, where O_i is the set of points at which all functions from the space L_i vanish;

(c) O contains the union of the pole divisors of all functions from the spaces $L_1, \dots, L_n$;

(d) the inequality $\dim O < \dim X$ holds.

By the well-definedness of the intersection index we mean that, *for a generic element* $(f_1, \dots, f_n) \in L_1 \times \dots \times L_n$, *all roots of the system of equations* $f_1 = \dots = f_n = 0$ *in the set* $X \setminus O$ *are nondegenerate* (*i.e., the differentials* df_i *are linearly independent at each root*), *and their number is independent of the choice of the set* O *satisfying conditions* (*a*)–(*d*) *and equals* $[L_1, \dots, L_n]$.

(2) Intersection index is symmetric, i.e., it is invariant under permutations of spaces. This property follows directly from the definition.

(3) The intersection index is multilinear. The linearity of the intersection index in the first argument means that, for any spaces $L_1' L_1'', L_2, \dots, L_n \in \mathscr{K}(X)$, the relation $[L_1', L_1'', L_2, \dots, L_n] = [L_1', L_2, \dots, L_n] + [L_1'', L_2, \dots, L_n]$ holds. Linearity in the other arguments is defined similarly.

2. The Grothendieck group of the semigroup $\mathscr{K}(X)$. As for any commutative semigroup, for the semigroup $\mathscr{K}(X)$, its *Grothendieck semigroup* and *Grothendieck group* are defined.

Definition 1. Two elements a and b of a commutative semigroup S are called *equivalent*, $a \sim b$, if there exists an element $c \in S$ such that $ac = bc$. The set of equivalence classes with induced multiplication forms a commutative *cancellative semigroup* (i.e., a semigroup in which $AC = BC$ implies $A = B$), called the *Grothendieck semigroup* of the semigroup S. We denote the Grothendieck semigroup of the semigroup $\mathscr{K}(X)$ by $\mathscr{K}_G(X)$.

Definition 2. With any commutative semigroup S one associates the set $G(S)$ of pairs (a, b) of elements a and b of its Grothendieck semigroup under the identification $(a, b) \sim (c, d) \iff ad = bc$. The set $G(S)$ with the operations $(a, b)(c, d) = (ac, bd)$ of multiplication and $(a, b)^{-1} = (b, a)$

of inversion forms a group, called the *Grothendieck group* of the semigroup S. We denote the Grothendieck group of the semigroup $\mathscr{K}(X)$ by $G(K)$.

Every homomorphism $\tau\colon S \to G$ of a commutative semigroup S to a commutative group G factors uniquely through the natural mapping $\rho\colon S \to G(S)$ of the semigroup S to its Grothendieck group; i.e., there is a unique homomorphism $\tilde{\tau}\colon G(S) \to G$ such that $\tau = \tilde{\tau} \circ \rho$.

Claim 1. *Let $L_1, \dots, L_n \in \mathscr{K}(X)$, and let $L'_1, \dots, L'_n \in \mathscr{K}(X)$. Suppose that $L_1 \sim L'_1, \dots, L_n \sim L'_n$. Then $[L_1, \dots, L_n] = [L'_1, \dots, L'_n]$.*

Proof. Let us show that $[L_1, L_2, \dots, L_n] = [L'_1, L_2, \dots, L_n]$. Indeed, the multilinearity of the intersection index implies that the mapping $\mathscr{K}(X) \mapsto \mathbb{Z}$ sending an element $L \in \mathscr{K}(X)$ to the number $[L, L_2, \dots, L_n] \in \mathbb{Z}$ is a homomorphism of the semigroup $\mathscr{K}(X)$ into the group $\mathbb{Z}$. This homomorphism can be extended to the Grothendieck group, which implies the required equality. By using the symmetry of the intersection index, one can replace the other spaces by equivalent ones without changing the intersection index.

Below we discuss some obvious properties of the equivalence relation $L_1 \sim L_2$ on the semigroup $\mathscr{K}(X)$.

Claim 2. (i) *If $L_1 \sim L_2$, then $L_1 + L_2 \sim L_1 \sim L_2$.*

(ii) *If $L_1 \sim L_2$ and the inclusions $L_1 \subset L \subset L_2$ hold, then $L_1 \sim L \sim L_2$.*

Proof. (i) If $L_1M = L_2M$, then $(L_1 + L_2)M = L_1M + L_2M = L_1M = L_2M$.

(ii) If $L_1 \subset L \subset L_2$ and $L_1M = L_2M$, then $L_1M \subset LM \subset L_2M$. Hence $L_1M = LM = L_2M$.

We say that a function $f \in \mathbb{C}(X)$ *is trivial over* $L \in \mathscr{K}(X)$ if $L \sim L(f)$, where $L(f)$ is the space spanned by the functions in L and the function f.

Claim 3. (i) *If $L \subset M$, $L \sim M$, and $f \in M$, then $L(f) \sim L$.*

(ii) *All functions trivial over $L \in \mathscr{K}(X)$ form a linear space over $\mathbb{C}$.*

(iii) *If $L(f) \sim L$ and g is trivial over $L(f)$, then $L(g) \sim L$.*

(iv) *If f is trivial over L^k and g is trivial over L^m, then fg is trivial over L^{k+m}.*

Proof. (i) We have $L \subset L(f) \subset M$. Since $L \sim M$, it follows that $L(f) \sim L$.

(ii) Suppose that $L(f) \sim L$ and $L(g) \sim L$. Then $L(f) + L(g) \sim L$. Since $L \subset L(\lambda f + \mu g) \subset L(f) + L(g)$ for any $\lambda, \mu \in \mathbb{C}$, it follows that $L(\lambda f + \mu g) \sim L$.

(iii) Let $L' = L(f)$. By assumption $L' \sim L$ and $L'(g) \sim L'$. Hence $L'(g) \sim L$, and therefore $L(g) \sim L$.

(iv) If $L^k(f) \sim L^k$ and $L^m(g) \sim L^m$, then $L^k(f)L^m(g) \sim L^{k+m}$. Since $L^{k+m} \subset L^k(f)L^m(g)$ and $fg \in L^k(f)L^m(g)$, it follows that $L^{k+m}(fg) \sim L^{k+m}$.

The triviality of a function f over a space L can be described in quite different terms (see [7] and [12]).

Definition 3. A function $f \in \mathbb{C}(X)$ is said to be *integral over* $L \in \mathscr{K}(X)$ if it satisfies an equation of the form

$$f^d + a_1 f^{d-1} + \dots + a_d = 0 \tag{1}$$

in which every coefficient a_i belongs to the space L^i.

Claim 4. *$L(f) \sim L$ if and only if the function f is integral over L.*

Proof. If f satisfies Eq. (1), then $L(f)(L(f))^d = L(L(f))^d$. This proves that a function integral over L is trivial over L.

Suppose that $L(f) \sim L$. Then there exists an $M \in \mathscr{K}(X)$ such that $LM = L(f)M$. Let $e_1, \dots, e_d$ be a basis of the space M over the field $\mathbb{C}$. The relation $LM = L(f)M$ implies $fe_i = \sum b_{i,j}e_j$ for some $b_{i,j}$ from the space L. Hence f is a root of the characteristic polynomial $\det(B - fI) = 0$ of the $(d \times d)$ matrix $B = \{b_{i,j}\}$. This shows that if $L(f) \sim L$, then f is integral over L.

Corollary 5. *Suppose that $L(f) \sim L$ and all functions in $L \in \mathscr{K}(X)$ are regular on a domain $U \subset X$. Then f is regular on U.*

3. Notation. In Sections 3.1 and 3.2 below we define special integer-valued functions in terms of which the bounds of Sections 4 and 5 are formulated. In Section 3.3 we introduce notation used in Sections 4 and 5.

3.1. Let $Q(n,l)$ be the dimension of the space of polynomials in n variables of degree $\leqslant l$. The number $Q(n,l)$ is equal to the number of integer points in the n-simplex Δ given by the inequalities $0 \leqslant u_1, \dots, 0 \leqslant u_n$, $u_1 + \dots + u_n \leqslant l$. The volume of the simplex Δ is equal to $l^n/n!$. Hence, for given n, we have $Q(n,l) \approx l^n/n!$ as $l \to \infty$. It is easy to see that $Q(n,l) = \binom{l+n}{n}$.

The space of homogeneous polynomials of degree k in $n+1$ variables is isomorphic to the space of polynomials of degree $\leqslant k$ in n variables. It follows that $Q(n+1,k) - Q(n+1,k-1) = Q(n,k)$, and hence

$$Q(n+1,k) - Q(n+1,k-d) = Q(n,k) + Q(n,k-1) + \dots + Q(n,k-d+1).$$

3.2. In Section 5 we need the following definition.

Definition 4. Let $N = kd + r$, where r, $0 \leqslant r < d$, is the remainder after the division of the number N by d. Consider the function F of (n,d,N) defined by the formula

$$F(n,d,N) = rQ(n,k) + (d-r)Q(n,k-1).$$

For given n and d, we have $F(n,d,N) \approx dk^n/n!$, where $k = [N/d]$, as $N \to \infty$.

3.3. In Sections 4 and 5 we use the following notation:

- X^* is an n-dimensional complex-analytic variety;
- L is a finite-dimensional space of analytic functions on X^* containing the constants; we assume that L contains a set of functions $x_1, \dots, x_n$ such that the solution set of the system of equations $x_1 = \dots = x_n = 0$ on X^* contains a subset $Y = \{y_1, \dots, y_d\}$ of nondegenerate solutions (i.e., such that the differentials dx_i of the functions x_i for $i = 1, \dots, n$ are linearly independent at the points of Y);
- $\mathbf{x}\colon X^* \to \mathbb{C}^n$ is the mapping given by $\mathbf{x}(q) = (x_1(q), \dots, x_n(q))$;
- $\mathbf{x}_i^{-1}\colon U \to V_i$ is a local inverse of $\mathbf{x}$ satisfying the condition $\mathbf{x}_i^{-1}(0) = y_i$, where V_i is a neighborhood of the point y_i and U is a neighborhood of the point 0 (the same for all i, $1 \leqslant i \leqslant d$).

4. Bounding dimension from below. In the following simple Lemma 6 we use the notation introduced in Section 3.

Lemma 6. 1. *Suppose that $f \in L$ takes different values at the points $y_1, \dots, y_d$. Then f satisfies no equation of the form*

$$a_1(\mathbf{x})f^{d-1} + \dots + a_d(\mathbf{x}) = 0$$

in which $a_1, \dots, a_d$ are polynomials on $\mathbb{C}^n$ (not vanishing simultaneously).

2. *If there exists a function $f \in L$ taking different values at the points $y_1, \dots, y_d$, then*

$$\dim_{\mathbb{C}} L^k \geqslant Q(n+1,k) - Q(n+1,k-d) = Q(n,k) + Q(n,k-1) + \dots + Q(n,k-d+1).$$

Proof. 1. Let f_i be the function defined in the neighborhood U by the formula $f_i = f(\mathbf{x}_i^{-1})$. For $i \neq j$, the functions f_i and f_j do not coincide in U and satisfy the equation $a_1 y^{d-1} + \dots + a_d = 0$. But an equation of degree $d-1$ with nonzero coefficients cannot have more than $d-1$ roots. This contradiction proves the first assertion.

2. The space L^k contains the functions $a_1(\mathbf{x})f^{d-1} + \dots + a_d(\mathbf{x})$, where the a_i are polynomials in $x_1, \dots, x_n$ of degree not exceeding $k-d+i$. According to the first assertion, these functions are linearly independent. This proves Lemma 6.

We say that a space $L \in \mathscr{K}(X)$ *separates the generic points* of an algebraic variety X if there exists a semialgebraic set $O(L) \subset X$ such that (i) $\dim O(L) < \dim X$; (ii) $O(L)$ contains all pole divisors of functions from L; (iii) for any two different points $a, b \in X \setminus O(L)$, there exists a function $g \in L$ such that $g(a) \neq g(b)$.

Lemma 7. *Suppose that $L \in \mathscr{K}(X)$ separates the generic points of X and $Y \subset X$ is a finite set disjoint from $O(L)$. Then there exists a function $f \in L$ taking different values at different points of Y.*

Proof. The set of functions in L taking the same value at two distinct points of Y is a proper subspace of L. The union of finitely many proper subspaces cannot coincide with L.

Theorem 8. *Suppose that $L \in \mathscr{K}(X)$ separates the generic points of an n-dimensional variety X and $[L, \dots, L] = d > 0$. Then*

$$\dim_{\mathbb{C}} L^k \geqslant Q(n+1, k) - Q(n+1, k-d) = Q(n, k) + Q(n, k-1) + \dots + Q(n, k-d+1).$$

Proof. Let $X^* = X \setminus (D \cup O)$, where D is the divisor of poles of functions from L and O is the union of the singular points of X with the set of common zeros of all functions from L. By Lemma 7 there is a function $f \in L$ taking different values at different points $y_1, \dots, y_d$. To complete the proof, it remains to apply assertion 2 of Lemma 6.

5. Bounding dimension from above. In the formulation of Lemma 9 we use notation introduced in Section 3 and *assume that the variety X^* is connected*.

Lemma 9. *Let M be a linear space of analytic functions on X^* such that it contains constants and $\dim_{\mathbb{C}} M > F(d, n, N)$. Then there are functions $l_1, \dots, l_{n-1} \in L$ and $\varphi \in M$ for which the system of equations $l_1 = \dots = l_{n-1} = \varphi = 0$ has at least N nondegenerate solutions in X^*.*

Proof. Take r points $y_1, \dots, y_r$ in the set Y. Let $\Omega_{k,Y}$ be the linear space each of whose elements is a set of k-jets of smooth functions at the points $y_1, \dots, y_r$ and a set of $(k-1)$-jets of smooth functions at the points $y_{r+1}, \dots, y_d$. It is clear that $\dim_{\mathbb{C}} \Omega_{k,Y} = F(n, d, N) < \dim_{\mathbb{C}} M$. Hence there exists a nonzero function $g \in M$ that goes to zero under the mapping $\tau \colon M \to \Omega_{k,Y}$ sending each function to the set of its k-jets at $y_1, \dots, y_r$ and the set of its $(k-1)$-jets at $y_{r+1}, \dots, y_d$.

If h is a homogeneous linear function on $\mathbb{C}^n$, then the function $h(\mathbf{x})$ belongs to L. Consider the homogeneous system of linear equations $h_1 = \dots = h_{n-1} = 0$ in $\mathbb{C}^n$. To this system there corresponds the system $l_1 = \dots = l_{n-1} = 0$ on X^*, where $l_i = h_i(\mathbf{x})$. This system determines smooth curves $\Gamma_1, \dots, \Gamma_d$ in the domains $V_1, \dots, V_d$ such these curves pass through the points $y_1, \dots, y_d$.

If the equations $h_1 = \dots = h_{n-1} = 0$ are generic enough, then the restrictions of the function g to the curves $\Gamma_1, \dots, \Gamma_d$ are not identically zero and have zeros of multiplicities $\geqslant k+1$ at $y_1, \dots, y_r$ and zeros of multiplicities $\geqslant k$ at $y_{r+1}, \dots, y_d$. Indeed, the mapping $\mathbf{x}$ identifies the neighborhoods $V_i \subset X^*$ of the points y_i with the neighborhood $U \subset \mathbb{C}^n$ of the point 0. This identification transforms each curve $\Gamma_1, \dots, \Gamma_d$ into the part of the line $h_1 = \dots = h_{n-1} = 0$ contained in the neighborhood U. The restriction of the function g to the domain V_i is identified with the function $g_i = g(\mathbf{x}_i^{-1})$ not vanishing identically in the neighborhood U (otherwise the analyticity of the function g and the connectedness of the variety X^* would imply that g vanishes identically as well, but is does not). Since the functions $g_1, \dots, g_d$ do not vanish identically on the domain U, it follows that the restriction of each function g_i to almost every line $h_1 = \dots = h_{n-1} = 0$ does not vanish identically.

Hence the system of equations $l_1 = \dots = l_{n-1} = \varphi = 0$, where $\varphi = g - \varepsilon = 0$, has at least N nondegenerate roots for small ε: at least $k+1$ nondegenerate roots on each of the curves $\Gamma_1, \dots, \Gamma_r$ and at least k nondegenerate roots on each of the curves $\Gamma_{r+1}, \dots, \Gamma_d$. This proves Lemma 9, since $(k+1)r + k(d-r) = kd + r = N$.

Theorem 10. *Let X be an irreducible n-dimensional algebraic variety such that $L \in \mathscr{K}(X)$ and $[L, \dots, L] = d > 0$. If $M \in \mathscr{K}(X)$ and $[L, \dots, L, M] \leqslant N$, then $\dim_{\mathbb{C}} M \leqslant F(n, d, N+1)$.*

Proof. Let $X^* = X \setminus (D \cup O)$, where D is the divisor of poles of the functions from L and M and O is the union of singular points of X with the sets O_1 and O_2 of common zeros of the functions from L and M, respectively. If $\dim M > F(n, d, N+1)$, then by Lemma 9 one can find functions $l_1, \dots, l_{n-1} \in L$ and $\phi \in M$ such that the system $l_1 = \dots = l_{n-1} = \phi = 0$ has at least

$N+1$ nondegenerate roots in X^*. This contradicts the inequality $[L,\dots,L,M]\leqslant N$ and proves Theorem 10.

Corollary 11. *Under the assumptions of Theorem* 10,

(i) *if* $M\sim L$, *then* $\dim_{\mathbb{C}} M\leqslant n+d$;

(ii) *if* $M\sim L^k$, *then* $\dim_{\mathbb{C}} M\leqslant Q(n,k+1)+(d-1)Q(n,k)$.

Proof. Assertion (ii) follows from Theorem 10, since $[L,\dots,L,L^k]=kd$ and $F(n,d,kd+1)=Q(n,k)+(d-1)Q(n,k-1)$. Assertion (i) follows from (ii), since $Q(n,1)=n+1$ and $Q(n,0)=1$.

6. Integral closure of a subspace. We revisit the equivalence turning the semigroup $\mathscr{K}(X)$ into its Grothendieck semigroup $\mathscr{K}_G(X)$.

Definition 5. For every space $L\in\mathscr{K}(X)$, we define its *integral closure* $\overline{L}$ to be the set of all functions $f\in\mathbb{C}(X)$ integral over L.

It follows from Claims 3 and 4 that the set $\overline{L}$ is a linear space. If $L,M\in\mathscr{K}(X)$ and $L\subset M$, then $\overline{L}\subset\overline{M}$.

Claim 12. *Let X be an irreducible n-dimesional algebraic variety such that $L\in\mathscr{K}(X)$ and $[L,\dots,L]=d>0$. Then $\dim_{\mathbb{C}}\overline{L}\leqslant d+n$.*

Proof. The required assertion follows from Corollary 11, (ii).

Corollary 13. (i) *If $L\in\mathscr{K}(X)$, then the dimension of $\overline{L}$ is finite, i.e., $\overline{L}\in\mathscr{K}(X)$.*

(ii) *If $M\in\mathscr{K}(X)$, then $M\sim L\iff\overline{M}=\overline{L}$.*

Proof. Assertion (i) in the case $[L,\dots,L]>0$ was proved in Claim 12. If $[L,\dots,L]=0$, then, instead of L, we take a larger space $L\subset M$ such that $[M,\dots,M]>0$. According to Claim 12, $\dim_{\mathbb{C}}\overline{M}<\infty$. This proves (i), since $\overline{L}\subset\overline{M}$.

(ii) According to (i), among all spaces equivalent to a given space L, there is a largest space $\overline{L}$. This proves (ii).

Corollary 13 can also be deduced from Noether's theorem on integral closure (see [7] and [12]). However, the bounds obtained above allow us to avoid appealing to algebraic geometry. Example 1 shows that the bound of Claim 12 is sharp.

Example 1. Let X be $\mathbb{C}^n$, and let $x_1,\dots,x_n$ be coordinates on $\mathbb{C}^n$. Take L to be the space generated by the functions $1,x_1,\dots,x_1^d$ and $x_2,\dots,x_n$. Then $[L,\dots,L]=d$ and $\dim_{\mathbb{C}} L=d+n$.

Definition 6. The *Hilbert function* H_L and the *normalized Hilbert function* $\overline{H}_L$ of a space $L\in\mathscr{K}(X)$ are defined as

$$H_L(k)=\dim_{\mathbb{C}} L^k,\qquad \overline{H}_L(k)=\dim_{\mathbb{C}}\overline{L^k}.$$

7. The case where the function space separates points.

Theorem 14. *Suppose that $L\in\mathscr{K}(X)$ separates the points of an irreducible n-dimensional algebraic variety X. Suppose also that $[L,\dots,L]=d>0$. Then*

$$\sum_{k-d<i\leqslant k}Q(n,i)\leqslant H_L(k)\leqslant\overline{H}_L(k)\leqslant F(n,d,kd+1),\tag{2}$$

$$[L,\dots,L]=\lim_{k\to\infty}\frac{n!\,H_L(k)}{k^n}=\lim_{k\to\infty}\frac{n!\,\overline{H}_L(k)}{k^n}\,.\tag{3}$$

Proof. Inequalities (2) follow from Theorems 8 and 10. We have $\sum_{k-d<i\leqslant k}Q(n,i)\approx F(n,d,kd+1)\approx dk^n/n!$. Therefore, (3) follows from (2).

Example 2. The lower bound $\sum_{k-d<i\leqslant k}Q(n,i)$ for $H_L(k)$ is sharp in the case where X is an algebraic hypersurface in $\mathbb{C}^{n+1}$ of degree d and the space L is generated by the coordinate functions in $\mathbb{C}^{n+1}$ and the constants.

Example 3. Let X be an irreducible plane algebraic curve of degree d, and let L be the space generated by the constants and the coordinate functions. In this case, the lower bound for the dimension $\dim_{\mathbb{C}} L^k$ is sharp (see Example 2). The upper bound $kd+1$ for $\overline{H}_L(k)$ is sharp for a curve X of genus zero; i.e., for such X, we have $\overline{H}_L(k) = kd + 1$. For $k \gg 0$ and any m satisfying $1 - (d-1)(d-2)/2 \leqslant m \leqslant 1$, one can construct a plane curve for which the lower bound is sharp, while the upper bound is $\dim_{\mathbb{C}} \overline{L^k} = kd + m$.

Example 4. If $[L, \dots, L] = 1$ or $[L, \dots, L] = 2$, then the upper bound for the function $\overline{H}_L(k)$ coincides with the lower bound for the function $H_L(k)$. In this case, the theorem gives a formula for $\overline{H}_L(k) = H_L(k)$.

The *degree* $d(X)$ of a projective n-dimensional variety $X \subset \mathbb{C}P^N$ is the number of its points of intersection with a generic linear subspace of codimension n. A variety $X \subset \mathbb{C}P^N$ can be associated with the ideal $I_X \subset \mathbb{C}[x_0, \dots, x_N]$ of polynomials in homogeneous coordinates $(x_0 : \dots : x_N)$ on $\mathbb{C}P^N$ vanishing on X and the ring $A_X = \mathbb{C}[x_0, \dots, x_N]/I_X$. The ring A_X is the direct sum of its homogeneous components: $A_X = A_0 + A_1 + \dots$. The *Hilbert function* $H_{[X]}$ of the variety $X \subset \mathbb{C}P^N$ is the function defined on the nonnegative integers by $H_{[X]}(k) = \dim_{\mathbb{C}} A_k$.

Hilbert's theorem. *For every irreducible n-dimensional projective variety $X \subset \mathbb{C}P^N$, the limit $l(X) = \lim_{k\to\infty} H_{[X]}(k)/k^n$ exists. Moreover, the degree $d(X)$ of the variety X is equal to $n!\, l(X)$.*

Remarks. The formulation of Hilbert's theorem usually contains also the statement that the function H_X is polynomial for large values of the argument. This property of the Hilbert function is valid for any finitely generated graded module over the ring of polynomials and is not related to degrees of projective varieties. In Hilbert's theorem, the assumtion that the variety X is irreducible can be dispensed with (the general case reduces to the case of an irreducible variety).

Proof of Hilbert's theorem. Theorem 14 not only contains Hilbert's theorem, but also provides explicit bounds for the value of the Hilbert function at any value of the argument. Indeed, let D be a generic hyperplane section. Let us represent $\mathbb{C}P^N$ as the union of $\mathbb{C}^N$ and the "hyperplane at infinity" $\mathbb{C}P^{N-1}$; we regard D as the intersection of the variety X with $\mathbb{C}P^{N-1}$.

Consider the affine variety $X_{\text{af}} = X \setminus D \subset \mathbb{C}^N$. Take the finite-dimensional space L of functions on X consisting of the restrictions to X_{af} of polynomials of degree 1. It follows directly from the definition that $H_L = H_{[X]}$. The degree $d(X)$ of the variety X is equal to the number of solutions of a generic system of equations $l_1 = \dots = l_n = 0$ on X_{af}, i.e., $d(X) = [L, \dots, L]$.

The variety X_{af}, together with the space of functions L, satisfies all conditions of Theorem 14. Indeed, $X_{\text{af}} \subset \mathbb{C}^N$, and L contains the restrictions to X_{af} of all coordinate functions on $\mathbb{C}^N$. Hence the space L separates the points of X_{af}. Let us show that $[L, \dots, L] > 0$. Take any affine subspace M of codimension n passing through some smooth point $a \in X_{\text{af}}$ transversally to X_{af}. The space M is given by a system of equations $l_1 = \dots = l_n = 0$, where the l_i are polynomials of degree 1. This system has a nondegenerate root a on X_{af}. A generic system of equations $l_1 = \dots = l_n = 0$ on X_{af} with $l_i \in L$ has at least the same number of roots on X_{af}. Hence $[L, \dots, L] > 0$. Now the lower and upper bounds for the function $H_{[X]}$ and Hilbert's theorem follow from Theorem 14.

8. The case of positive self-intersection.

Claim 15. *Suppose that X is an irreducible n-dimensional algebraic variety, $L \in \mathscr{K}(X)$, and $[L, \dots, L] = d > 0$. Consider the mapping $\mathbf{x}\colon X \to \mathbb{C}^n$ such that $\mathbf{x} = (x_1, \dots, x_n)$ is a generic set of functions from the space L. Then every function $f \in \mathbb{C}(X)$ satisfies an equation of the form $f^d + a_1(\mathbf{x}) f^{d-1} + \dots + a_0(\mathbf{x}) = 0$ for some rational functions $a_1, \dots, a_d$ on $\mathbb{C}^n$.*

Proof. Since $[L, \dots, L] = d$, it follows that there is a semialgebraic set $\Sigma \subset \mathbb{C}^n$ in $\mathbb{C}^n$ such that $\dim \Sigma < n$ and every point $z \in \mathbb{C}^n \setminus \Sigma$ has exactly d nondegenerate preimages $y_i = \mathbf{x}_i^{-1}(z)$ under the mapping $\mathbf{x}\colon X \to \mathbb{C}^n$. For $\phi \in \mathbb{C}(X)$, the function $\operatorname{Trace}_{\mathbf{x}} \phi$ defined on the domain $\mathbb{C}^n \setminus \Sigma$ by the formula $\operatorname{Trace}_{\mathbf{x}} \phi = \sum_i \phi(\mathbf{x}_i^{-1})$ can be extended to a rational function on $\mathbb{C}^n$. We denote this extension by the same symbol $\operatorname{Trace}_{\mathbf{x}} \phi$. For $k = 0, \dots, d-1$, consider the symmetric power sums $N_k = \sum_i f^k(\mathbf{x}_i^{-1}) = \operatorname{Trace}_{\mathbf{x}} f^k$ of the branches $f(\mathbf{x}_i^{-1})$ of the multivalued function $f(\mathbf{x}^{-1})$. Let $S_k = P_k(N_1, \dots, N_k)$ be the expressions of the elementary symmetric functions of the branches $f(\mathbf{x}_i^{-1})$ in terms of the symmetric power sums of these branches. Then $f^d - S_1 f^{d-1} + \dots + (-1)^d S_d = 0$.

Corollary 16. *Under the assumptions of Claim* 15 *there is a number* m *such that the space* $\overline{L^m}$ *contains a function* g *that separates the points* $y_1, \dots, y_d$.

Proof. Let z be a point in the domain $\mathbb{C}^n \setminus \Sigma$, and let $f \in \mathbb{C}(X)$ be a function regular at all points of the set $Y = \mathbf{x}^{-1}(z)$ and taking different values at different points of this set. The function f satisfies an equation of the form $f^d - S_1 f^{d-1} + \dots + (-1)^d S_d = 0$, where the S_i are rational functions regular at the point z. If we let Q be the least common multiple of the denominators of the rational functions $S_1, \dots, S_d$, then the function $g = fQ$ satisfies the equation $g^d - QS_1 g^{d-1} + \dots + (-1)^d Q^d S_d = 0$ with polynomial coefficients $(-1)^i Q^i S_i$. The function g separates the points of Y, since $Q(z) \neq 0$ and the function f separates the points of Y. It remains to note that g belongs to the space $\overline{L^m}$, where m is any number greater than all numbers $\deg(Q^i S_i)/i$ for $i = 1, \dots, d$.

Theorem 17. *Suppose that* X *is an irreducible* n*-dimensional algebraic variety,* $L \in \mathscr{K}(X)$, *and* $[L, \dots, L] = d > 0$. *Then*

$$\overline{H}_L(k) \leqslant F(n, d, kd + 1), \tag{4}$$

$$[L, \dots, L] = \lim_{k \to \infty} \frac{n!\, \overline{H}_L(k)}{k^n}. \tag{5}$$

Proof. Inequality (4) follows from Theorem 10. By Corollary 16 the space $\overline{L^m}$ separates the generic points of X for some m. We have $[\overline{L^m}, \dots, \overline{L^m}] = dm^n$. Let $p = [k/m]$. Since $(\overline{L^m})^p \subset \overline{L^k}$, it follows from the lower bound in (2) that $\dim_{\mathbb{C}} \overline{L^k} \geqslant \dim_{\mathbb{C}} (\overline{L^m})^p \geqslant \sum_{p - dn^m < i \leqslant p} Q(n, i) \approx dm^n p^n/n! \approx dk^n/n!$. To prove (5), it is sufficient to use inequality (4), since $F(n, d, kd+1) \approx dk^n/n!$.

9. General case. With any space $L \in \mathbb{C}(X)$ the rational mapping $\rho_L \colon X \to L^*$ defined by the formula $\langle \rho_L(x), f \rangle = f(x)$ for $f \in L$ is associated. It is called the *generalized Veronese mapping.* Let X_L denote the Zariski closure of the image of the variety X under the mapping ρ_L.

Claim 18. *Let* X *be an irreducible* n*-dimensional algebraic variety, and let* $L \in \mathscr{K}(X)$. *Then* $[L, \dots, L] > 0$ *if and only if* $\dim X_L = n$.

Proof. If $\dim X_L < n$, then almost every affine subspace of codimension n in L^* is disjoint from the variety X_L. Hence almost every system of equations $l_1 = \dots = l_n = 0$ with $l_i \in L$ has no solutions in X, i.e., $[L, \dots, L] = 0$.

Suppose that $\dim X_L = n$. Let $X^* = X \setminus O$, where O is the union of the singular locus of X with the divisor of poles of the functions from the space L. The image of the variety X^* under the generalized Veronese mapping has dimension n. Hence one can find an affine subspace of codimension n intersecting this image transversally. This subspace corresponds to a system of equations $l_1 = \dots = l_n = 0$ with $l_i \in L$ having nondegenerate solutions in X^*; hence $[L, \dots, L] > 0$.

With the mapping $\rho_L \colon X \to X_L$ we associate the algebraic variety $\widetilde{X}_L$ defined up to birational isomorphism as follows. Let $\mathbb{C}(\widetilde{X}_L)$ be the subfield of the field $\mathbb{C}(X)$ consisting of rational functions on the variety X which are constant on each irreducible component of the preimage $\rho^{-1}(z) \subset X$ of every point $z \in X_L$.

The mapping $\rho_L^* \colon \mathbb{C}(X_L) \to \mathbb{C}(X)$ determines an embedding of the field $\mathbb{C}(X_L)$ into the field $\mathbb{C}(\widetilde{X}_L) \subset \mathbb{C}(X)$. The field $\mathbb{C}(\widetilde{X}_L)$ is a finite extension of its subfield $\rho_L^*(\mathbb{C}(X_L))$, since the numbers of irreducible components in the preimages $\rho^{-1}(z) \subset X$ are bounded by a constant (independent of the point $z \in X_L$).

Definition 7. We define $\widetilde{X}_L$ as an algebraic variety whose field of rational functions is isomorphic to the field $\mathbb{C}(\widetilde{X}_L)$.

The embeddings $\mathbb{C}(\widetilde{X}_L) \subset \mathbb{C}(X)$ and $\rho_L^* \colon \mathbb{C}(X_L) \to \mathbb{C}(\widetilde{X}_L)$ induce mappings $\widetilde{\pi}_L \colon X \to \widetilde{X}_L$ and $\widetilde{\rho}_L \colon \widetilde{X}_L \to X_L$. It follows from the definition that $\widetilde{\rho}_L$ has finite degree and $\dim X_L = \dim \widetilde{X}_L$. If the mapping $\rho_L \colon X \to X_L$ has finite degree, then the varieties $\widetilde{X}_L$ and X are birationally isomorphic.

Claim 19. *If a function $f \in \mathbb{C}(X)$ satisfies an algebraic equation over the subfield $\rho_L^*(\mathbb{C}(X_L)) \subset \mathbb{C}(X)$, then $f \in \mathbb{C}(\widetilde{X}_L)$.*

Proof. If $f^k + \rho_L^*(a_1)f^{k-1} + \dots + \rho_L^*(a_k) = 0$, where $a_i \in \mathbb{C}(X_L)$, then the function f is constant on each irreducible component of the preimage $\rho^{-1}(z) \subset X$ of every point $z \in X_L$.

Theorem 20. *Suppose that X is an irreducible algebraic variety, $L \in \mathscr{K}(X)$, $\dim X_L = p$, $\widetilde{L} = \rho_L^*(L) \in \mathscr{K}(\widetilde{X}_L)$, and $[\widetilde{L}, \dots, \widetilde{L}] = d$. Then*

(i) $d > 0$;

(ii) $\overline{H}_L(k) \leqslant F(p, d, kd + 1)$;

(iii) *if $\widetilde{\rho}_L \colon \widetilde{X}_L \to X_L$ is a birational isomorphism, then*

$$\sum_{k-d<i\leqslant k} Q(p,i) \leqslant H_L(k) \quad \text{and} \quad [\widetilde{L}, \dots, \widetilde{L}] = \lim_{k\to\infty} \frac{p!\, H_L(k)}{k^p} = \lim_{k\to\infty} \frac{p!\, \overline{H}_L(k)}{k^p};$$

(iv) $[\widetilde{L}, \dots, \widetilde{L}] = \lim_{k\to\infty} p!\, \overline{H}_L(k)/k^p$ *for any $L \in \mathscr{K}(X)$.*

Proof. The assumptions of the theorem imply $H_L(k) = \dim_{\mathbb{C}}(\widetilde{L})^k$ and $\overline{H}_L(k) = \dim_{\mathbb{C}} \overline{(\widetilde{L}^k)}$. Hence to prove the theorem, it suffices to consider the self-intersection index of the space $\widetilde{L}$ on the variety $\widetilde{X}_L$.

(i) The positivity of the self-intersection index d follows from the equality of dimensions $\dim \widetilde{X}_L = \dim X_L$ and Claim 18.

(ii) The required inequality follows from inequality (4) in Theorem 17 applied to the variety $\widetilde{X}_L$ and the space $\widetilde{L}$.

(iii) Under the assumptions of (iii) the space $\widetilde{L}$ separates the generic points of the variety $\widetilde{X}_L$. Hence (iii) follows from Theorem 14.

(iv) The required assertion follows from Theorem 17.

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