MOSCOW MATHEMATICAL JOURNAL
Volume 3, Number 3, July–September 2003, Pages 1013–1037

L-CONVEX-CONCAVE SETS IN REAL PROJECTIVE SPACE AND L-DUALITY

A. KHOVANSKII AND D. NOVIKOV

Dedicated to Vladimir Igorevich Arnold on the occasion of his 65th birthday

ABSTRACT. We define a class of L-convex-concave subsets of $\mathbb{R}P^n$, where L is a projective subspace of dimension l in $\mathbb{R}P^n$. These are sets whose sections by any $(l+1)$-dimensional space L' containing L are convex and concavely depend on L'. We introduce an L-duality for these sets and prove that the L-dual to an L-convex-concave set is an L^*-convex-concave subset of $(\mathbb{R}P^n)^*$. We discuss a version of Arnold's conjecture for these sets and prove that it is true (or false) for an L-convex-concave set and its L-dual simultaneously.

2000 MATH. SUBJ. CLASS. 52A30, 52A35.

KEY WORDS AND PHRASES. Separability, duality, convex-concave set, nondegenerate projective hypersurfaces.

INTRODUCTION

Convex-concave sets and Arnold's conjecture. The notion of convexity is usually defined for subsets of affine spaces, but it can be generalized to subsets of projective spaces. Namely, a subset of projective space $\mathbb{R}P^n$ is called *convex* if it doesn't intersect some hyperplane $L \subset \mathbb{R}P^n$ and is convex in the affine space $\mathbb{R}P^n \setminus L$. In the very definition of a convex subset of projective space, a hyperplane L appears. Projective space has subspaces L of different dimensions, not only hyperplanes. For any subspace L, we can define a class of L-convex-concave sets. These sets are the main object of investigation in this paper. If L is a hyperplane, then this class coincides with the class of closed convex sets lying in the affine chart $\mathbb{R}P^n \setminus L$. The definition of L-convex-concave sets is as follows.

A closed set $A \subset \mathbb{R}P^n$ is *L-convex-concave* if (i) the set A doesn't intersect the projective subspace L, (ii) for any $(\dim L + 1)$-dimensional subspace $N \subset \mathbb{R}P^n$ containing L, the section $A \cap N$ of the set A by N is convex, and (iii) for any

Received August 5, 2002.

The work of the first named author is supported in part by the Canadian Grant No. 0GP0156833. The work of the second named author is supported by the Killam grant of P. Milman, by the McDonnell Foundation and by NSF grant DMS 0200861.

$(\dim L - 1)$-dimensional subspace $T \subset L$, the complement to the projection of the set A from the center T on the factor-space $\mathbb{R}P^n/T$ is an open convex set.

Example. In projective space $\mathbb{R}P^n$ with homogeneous coordinates $x_0 : \cdots : x_n$, consider the set $A \subset \mathbb{R}P^n$ defined by the inequality $\{K(x) \leq 0\}$, where K is a non-degenerate quadratic form on $\mathbb{R}^{n+1}$. Suppose that K is positive definite on some $(k+1)$-dimensional subspace and negative definite on some $(n-k)$-dimensional subspace. In other words, suppose that K is of the form $K(x) = x_0^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2$ up to a linear change of coordinates. In this case, the set A is L-convex-concave with respect to the projectivization L of any $(k+1)$-dimensional subspace of $\mathbb{R}^{n+1}$ on which K is positive definite.

We are mainly interested in the following conjecture.

Main Conjecture. *Any L-convex-concave subset A of an n-dimensional projective space contains a projective subspace M of dimension $(n-1-\dim L)$.*

Note that any projective subspace of dimension larger than $(n - 1 - \dim L)$ necessarily intersects L, so it cannot be contained in A. For the quadratic set A from the above example, the main conjecture is evidently true: as M we can take the projectivization of any $(n-k)$-dimensional subspace of $\mathbb{R}^{n+1}$ on which K is negative definite.

For an L-convex-concave set A with a smooth non-degenerate boundary B, the main conjecture is a special case of the following conjecture of Arnold ([Ar1], [Ar2]).

Arnold's Conjecture. *Let $B \subset \mathbb{R}P^n$ be a connected smooth hypersurface bounding some domain $U \subset \mathbb{R}P^n$. Suppose that, at any point of B, the second fundamental form of B with respect to the outer normal vector is non-degenerate. Suppose that this form has (necessarily constant) signature $(n-k-1, k)$, i. e., at each point $b \in B$, the restriction of the second quadratic form to some k-dimensional subspace of T_bB is negative definite and its restriction to some $(n-k-1)$-dimensional subspace of T_bB is positive definite.*

Then there exist a projective subspace of dimension $(n-k-1)$ contained in the domain U and a projective subspace of dimension k in the complement $\mathbb{R}P^n \setminus \overline{U}$.

Our main conjecture and the very notion of L-convex-concavity arose from an attempt to prove or disprove the Arnold conjecture. We were not able to prove it in full generality. However, we obtained several results in this direction.

We proved the Arnold conjecture for hypersurfaces satisfying the following additional assumption: there exist a non-degenerate quadratic cone K and a hyperplane $\pi \subset \mathbb{R}P^n$ not passing through the vertex of the cone such that, first, the hypersurface and the cone K have the same intersection with the hyperplane π and, secondly, at each point of this intersection, the tangent planes to the hypersurface and to the cone coincide (the proof is given in [KhN2]).

There is the *affine* version of the Arnold conjecture, where $\mathbb{R}P^n$ is replaced by $\mathbb{R}^n$ (and the question is whether there exist affine subspaces of dimensions k and $(n-k-1)$ in U and $\mathbb{R}^n \setminus \overline{U}$, respectively). Our second result is an explicit construction of a counterexample to this affine version of Arnold's conjecture (see [KhN2]). The main role in this construction is played by *affine* convex-concave sets.

The class of (L)-convex-concave subsets of $\mathbb{R}^n$ is defined as follows. Fix the class (L) of $(k+1)$-dimensional affine subspaces of $\mathbb{R}^n$ parallel to L. Its elements are parameterized by points of the quotient space $\mathbb{R}^n/N$, where N is the (only) linear subspace from this class. A set A is called *affine (L-) convex-concave* if

(1) the section $A \cap N$ of A by any subspace $N \in (L)$ is convex and
(2) the section $A \cap N_a$ depends concavely on the parameter $a \in \mathbb{R}^n/N$.

The latter condition means that, for any segment $a_t = ta + (1-t)b$, where $0 \le t \le 1$, in the parameter space $\mathbb{R}^n/N$, the section $A \cap N_{a_t}$ is contained inside the linear combination (in the Minkowski sense) $t(A \cap N_a) + (1-t)(A \cap N_b)$ of the sections $A \cap N_a$ and $A \cap N_b$. Any projective L-convex-concave set is affine (L)-convex-concave in any affine chart not containing L with respect to the class (L) of the $(\dim L + 1)$-dimensional affine subspaces whose closures in $\mathbb{R}P^n$ contain L.

For the class (L) of planes parallel to L in $\mathbb{R}^3$, we constructed an (L)-convex-concave set $A \subset \mathbb{R}^3$ not containing lines with smooth and everywhere non-degenerate boundary, see [KhN2]. However, all our attempts to modify the example in such a way that its closure $\overline{A} \subset \mathbb{R}P^3$ be L-convex-concave failed. In the end, we have proved that this is impossible; to be more precise, the main conjecture is true for $\mathbb{R}^3$ and any L-convex-concave set with $\dim L = 1$ (see [KhN1]). This is the only case of the main conjecture which we were able to prove (except the trivially true cases of $\dim L = 0$ and $\dim L = n-1$ in projective space $\mathbb{R}P^n$ of any dimension n).

The main conjecture in the three-dimensional case. Our proof of the main conjecture in the three-dimensional case is quite lengthy. In this paper, we construct an L-duality needed for the fourth step of the proof (see the sketch of the proof below). The third step of the proof involves a cumbersome combinatorics, see [KhN1].

Below, we give a sketch of the proof and clarify the role of L-duality.

Sketch of the proof. Any line lying inside an L-convex-concave set $A \subset \mathbb{R}P^3$ intersects all convex sections $A \cap N$ of A by planes N containing the line L, and *vice versa*, any line intersecting all these sections lies in A. The *first step* of the proof is an application of a Helly theorem [He1], [He2]. Consider the four-dimensional affine space of all lines in $\mathbb{R}P^3$ not intersecting L; let U_N be the convex subsets of this space consisting of all lines intersecting the section $A \cap N$. Applying the Helly theorem to the family U_N, we conclude that if, for any five sections $A \cap N_i$, $i = 1, \ldots, 5$, there is a line intersecting all of them, then there is a line intersecting all the sections.

For any four sections, we can prove the existence of a line intersecting all of them. The *second step* consists in prooving this assertion (in any dimension). Namely, we have the following proposition.

Proposition 1 (about four sections)**.** *Let A be an L-convex-concave subset of $\mathbb{R}P^n$, and let $\dim L = n-2$. Then, for any four sections $A \cap N_i$ of the set A by hyperplanes N_i, where $i = 1, \ldots, 4$ and $N_i \supset L$, there exists a line intersecting all of them.*

The proof uses a theorem of Browder [Br]. This theorem is a version of the Brawer fixed point theorem asserting the existence of a fixed point for a continuous self-mapping of a closed n-dimensional ball. The Browder theorem deals with set-valued upper semi-continuous maps of a convex set B^n into the set of all its closed convex subsets. The Browder theorem claims that there is a point $a \in B^n$ such that $a \in f(a)$.

We use it as follows. Since the set $A \subset \mathbb{R}P^n$ is L-convex-concave and because of the assumption $\operatorname{codim} L = 2$, we easily deduce that, for any three sections $A_i = A \cap N_i$ with $i = 1, 2, 3$, and any point $a_1 \in A_1$, there is a line passing through a_1 and intersecting both A_2 and A_3. For the four sections $A_i = A \cap N_i$, where $i = 1, \dots, 4$, and a point $a_1 \in A_1$, consider all pairs of lines l_1 and l_2 such that

(1) the line l_1 passes through a_1 and intersects A_2 and A_3;
(2) the line l_2 passes through the intersection point of l_1 and A_3, intersects A_4, and intersects A_1 at a point a_1'.

Consider a set-valued mapping f of the section A_1 to the set of all its subsets which maps the point a_1 to the set of all points a_1' that can be obtained as specified above. We prove that f satisfies conditions of the Browder theorem. Hence there exists a point $a_1 \in A_1$ such that $a_1 \in f(a_1)$. This means that there is a line l_1 passing through this point and coinciding with the corresponding line l_2. Therefore, this line intersects the sections A_2, A_3, and A_4, which completes the second step of the proof.

The proof of the existence of a line intersecting the sections $A \cap N_i$, with $i = 1, \dots, 5$ (they are assumed to be fixed from now on) is quite complicated; the outline is as follows. Choose an affine chart containing all the five sections and not containing the line L. Fix a Euclidean metric in this chart.

We define the *distance* from a line l to the collection of sections $A \cap N_i$, where $i = 1, \dots, 5$, as the maximum distance from the point $a_i = l \cap N_i$ to the section $A \cap N_i$, over $i = 1, \dots, 5$. A line l is a *Chebyshev* line if the distance from l to the sections $A \cap N_i$, $i = 1, \dots, 5$, is minimal. We prove that, for the Chebyshev lines, these distances are all equal. With a Chebyshev line l, we associate five half-planes $p_i^+ \subset N_i$. These half-planes are supporting to the sections $A \cap N_i$ at the points $b_i \in A \cap N_i$ closest to a_i among all points of the section $A \cap N_i$. We have to prove that the distance from L to the sections is equal to zero, i. e., that $a_i = b_i$.

To prove this, it is sufficient to find a line l' intersecting all half-planes p_i^+ with $i = 1, \dots, 5$. Indeed, if $a_i \neq b_i$ then, slightly moving the line l in the direction of the line l', we can decrease the distance from the line l to the sections $A \cap N_i$, $i = 1, \dots, 5$, which is impossible. So, it is sufficient to prove that there exists a line l intersecting the five support half-planes $p_i^+ \subset N_i$, where $i = 1, \dots, 5$.

We say that the configuration of half-planes $p_i^+ \subset N_i$, $i = 1, \dots, 5$, is *non-degenerate* if their boundaries intersect the line L in five different points. Otherwise, i. e., if their boundaries intersect L in less than five points, we call the configuration *degenerate*. We prove the existence of the line l' separately for non-degenerate (step 3) and degenerate (step 4) configurations.

A detailed proof of the third step is given in [KhN1].

Below we give a brief sketch of this *third step*. The proof of the existence of a line intersecting all the five half-planes $p_i^+ \subset N_i$ of a non-degenerate configuration is based on a detailed analysis of combinatorial properties of each possible configuration. It turns out that there are essentially only six possible combinatorial types. For different combinatorial types of configurations, the proofs are different, though in the same spirit.

A rough description of the most general scheme is as follows. Instead of the half-planes $p_i^+ \subset N_i$, $i = 1, \ldots, 5$, consider extended half-planes p_i such that

(1) $p_i^+ \subset p_i \subset N_i$;
(2) the boundaries of the half-planes p_i intersect the Chebyshev line; and
(3) the intersections of the boundaries of p_i and p_i^+ with the line L coincide.

It is sufficient to prove that there exists a line intersecting all the extended half-planes $p_i \subset N_i$, $i = 1, \ldots, 5$, and at least one of the intersection points is interior. Take the planes π_i containing the Chebyshev line l and the boundaries of half-planes p_i for $i = 1, \ldots, 5$. Each half-plane p_j is divided by the planes π_i into five sectors. The minimizing property of the Chebyshev line l implies that some of the sectors necessarily intersect the convex-concave set A.

Using the combinatorial properties of the configuration, we choose four half-planes and a sector on one of them which intersects the set A. Applying the Browder theorem (as at step 2), we prove the existence of a line intersecting the four sections *in some prescribed sectors* of the corresponding half-planes. The combinatorial properties of the configuration imply that the constructed line intersects the fifth half-plane, q.e.d.

In the present paper, we prove, among the other things, the assertion of the *fourth step*, i. e., the existence of a line intersecting all the five half-planes $p_i^+ \subset N_i$ of a degenerate configuration. It is proved as follows. All hyperplanes $N \subset \mathbb{R}P^n$ containing a fixed subspace L of codimension 2 can be parameterized by points of the projective line $\mathbb{R}P^n/L$; so they carry a natural cyclic order. We say that an L-convex-concave set A with $\dim L = n-2$ is *linear between cyclically ordered sections* $A_i = A \cap N_i$ if the intersection A_{ij} of the set A with the half-space of the projective space bounded by the two adjacent hyperplanes N_i and N_{i+1} coincides with the convex hull of the sections $A_i = A \cap N_i$ and $A_{i+1} = A \cap N_{i+1}$ (the convex hull is taken in an arbitrary affine chart $\mathbb{R}P^n \setminus N_j$, where $j \neq i$ and $j \neq i+1$, and does not depend on the choice of the chart).

Proposition 2. *Let A be an L-convex-concave subset of* $\mathbb{R}P^n$, *where* $\dim L = n-2$. *Suppose that there exist four sections of the set A such that A is linear between these sections. Then the set A contains a line.*

This is a reformulation of Proposition 1.

We prove the following assertion, which is dual to Proposition 2.

Proposition 3 (about sets with octagonal sections)**.** *Let $D \subset \mathbb{R}P^n$ be an L-convex-concave set, where* $\dim L = 1$. *Suppose that any section $D \cap N$ of D by any two-dimensional plane N containing the line L is an octagon whose sides lie on lines intersecting the line L in four fixed* (*i. e., not depending on N*) *points*; *in other words, each octagon has four pairs of "parallel" sides intersecting L in a fixed point. Then there exists an $(n-2)$-dimensional projective subspace intersecting all planar sections $D \cap N$, where $L \subset N$, of the set D.*

In fact, the main goal of this paper is to give a definition of an L-duality with respect to which the two propositions above are dual and establish the general properties of this duality required to reduce Proposition 3 to Proposition 2.

Let us return to step 4 of the proof. In the cases of the degenerate configuration, the boundaries of the five half-planes p_i^+, $i = 1, \dots, 5$, intersect the line L in at most four points. We assume that their number is exactly four and denote them by Q_1, Q_2, Q_3, and Q_4. Now, consider the surgery of the set A which consists in replacing each convex section $A \cap N$ of the set A, where $L \subset N$, by a circumscribed octagon whose four pairs of parallel sides intersect the line L at the points $Q_1, \dots, Q_4$. In Section 6, we prove that the application of this surgery to an L-convex-concave set A gives an L-convex-concave set D. The set D satisfies the conditions of Proposition 3, so there exists a line intersecting all octagonal sections of the set D. This line intersects all the half-planes p_i^+, $i = 1, \dots, 5$, which completes the proof of the main conjecture in the three-dimensional case. □

L-duality and the plan of the paper. There are several well-known types of duality, e. g., the usual projective duality or the duality between the convex subsets of $\mathbb{R}^n$ containing the origin and the convex subsets of the dual space. Different types of duality are useful for different purposes. Here, we shall construct an L-duality, which maps an L-convex-concave subset A of projective space $\mathbb{R}P^n$ to a set $A_L^{\perp}$ in the dual projective space $(\mathbb{R}P^n)^*$. The set $A_L^{\perp}$ turns out to be L^*-convex-concave, where $L^* \subset (\mathbb{R}P^n)^*$ is the subspace dual to L. The L-duality has the main duality property holds, namely, $A = (A_L^{\perp})_{L^*}^{\perp}$. It turns out that the main conjecture holds for a set $A \subset \mathbb{R}P^n$ if and only if it holds for its dual $A_L^{\perp} \subset (\mathbb{R}P^n)^*$: if the set $A_L^{\perp}$ contains a projective subspace M^* such that $\dim M^* + \dim L^* = n - 1$, then the set A contains the dual subspace M such that $\dim M + \dim L = n - 1$. This is why L-duality is useful for us: the problem for the L-dual set may be simpler than that for the initial set. Such a situation occurs at step 4 of the proof of the main conjecture in the three-dimensional case.

In this paper, we give a detailed description of L-duality. Its geometric meaning is easy-to-understand if the L-convex-concave set A is a domain with a smooth boundary. Assume that the boundary B of A is strictly convex-concave, i. e., that its second quadratic form is non-degenerate at each point. Consider the hypersurface B^* in the dual projective space $(\mathbb{R}P^n)^*$ projectively dual (in the classical sense) to B. The smooth hypersurface B^* divides $(\mathbb{R}P^n)^*$ into two parts. The subspace L^* dual to L does not intersect the hypersurface B^*, so exactly one of the connected components of $(\mathbb{R}P^n)^* \setminus B^*$ does not contain L^*. The L-dual of the set A coincides with the closure of this component.

This definition does not work for sets whose boundaries are not smooth and strictly convex-concave. However, we have to deal with precisely such sets (in particular, with sets whose sections are closed convex polygons and whose complements to projections are open convex polygons). Therefore, we must give a different, more suitable to our settings, definition. An example of how duality can be defined in such a general situation is the classical definition of dual convex sets. We follow closely this example.

The paper is organized as follows. First, in Section 1, we give a definition of projective separability mimicking the standard definition of separability for affine spaces. All the statements formulated in this section are immediate, so we omit the proofs. In Section 2 we discuss projective duality, the notion mimicking the classical definition of duality for convex subsets of linear spaces containing the origin. The statements are also very simple, but for the sake of completeness, we give their proofs and explain why all of them are analogues of classical ones.

After that, in Section 3, we define L-duality and prove its basic properties (using the already defined notions of projective separability and projective duality). At the end of Section 3 we discuss semi-algebraic L-convex-concave sets and a relation between L-duality and integration along the Euler characteristic. The results of Sections 5 and 6 are used at step 4 of the proof of the main conjecture in the three-dimensional case. The results of Section 4 imply, in particular, the proposition about convex-concave sets with octagonal sections (the Proposition 3 above). In Section 6, we describe, in particular, the surgery allowing to circumscribe convex octagons about planar convex sections.

1. Projective and Affine Separability

We recall the terminology related to the notion of separability in projective and affine spaces.

Projective case. We say that a subset $A \subset \mathbb{R}P^n$ is *projectively separable* if any point of its complement lies on a hyperplane not intersecting the set A.

Proposition. *The complement to a projectively separable set A coincides with the union of all hyperplanes disjoint from the set A. Vice versa, the complement to any union of hyperplanes has the property of projective separability.*

This proposition can be reformulatedas follows.

Proposition. *Any subset of projective space defined by a system of linear homogeneous inequalities $L_\alpha \neq 0$, where α belongs to some index set and L_α is a homogeneous polynomial of degree one, is projectively separable. Vice versa, any projectively separable set can be defined in this way.*

We define the *projectively separable hull of the set A* as the smallest projectively separable set containing the set A.

Proposition. *The projectively separable hull of a set A is exactly the complement to the union of all hyperplanes in $\mathbb{R}P^n$ disjoint from A. In other words, a point lies in the projectively separable hull of a set A if and only if any hyperplane containing this point intersects the set A.*

Affine case. Recall the well-known notion of separability in the affine case. Namely, a subset A of an affine space is *affinely separable* if any point of the complement to the set A belongs to a closed half-space not intersecting the set A. Evidently, any affinely separable set is convex and connected.

Proposition. *The complement to an affinely separable set A coincides with the union of closed half-spaces not intersecting the set A. Vice versa, a complement to any union of closed half-spaces is affinely separable.*

This property can be reformulated as follows.

Proposition. *Any subset of an affine space defined by a system of linear inequalities $\{L_\alpha(x) < 0\}$, where α belongs to some index set and L_α is a polynomial of degree at most one, is affinely separable. Vice versa, any affinely separable set can be defined in this way.*

We define an *affinely separable hull* of a set A as the smallest set containing the set A and having the property of affine separability.

Proposition. *The affinely separable hull of a set A is equal to the complement to the union of all closed half-spaces of the affine space disjoint from A. In other words, a point lies in the affinely separable hull of the set A if and only if any closed half-space containing this point also intersects the set A.*

Convex subsets of projective spaces and separability. Projective and affine separability are closely connected.

Proposition. *Let L be a hyperplane in projective space $\mathbb{R}P^n$, and let $U = \mathbb{R}P^n \setminus L$ be the corresponding affine chart.*

1. *Any affinely separable subset of the affine chart U (which are, in particular, connected and convex in U) is also projectively separable as a subset of a projective space.*
2. *Any connected projectively separable subset of the affine chart U is also affinely separable as a subset of the affine space U.*

A connected projectively separable subset of projective space disjoint from at least one hyperplane is called a *separable convex* subset of projective space. (There is exactly one projectively separable subset of projective space intersecting all hyperplanes, namely, the projective space itself.)

Remark. We have defined above the notion of a (not necessarily projectively separable) convex subset of a projective space: a nonempty subset A of projective space $\mathbb{R}P^n$ is called *convex* if, first, there is a hyperplane $L \subset \mathbb{R}P^n$ disjoint from the set A and, secondly, any two points of the set A can be joined by a segment lying in A. We will not need convex non-separable sets.

2. Projective and Linear Duality

In this section, we construct a variant of projective duality. To a subset A of projective space $\mathbb{R}P^n$, it assigns a subset A_p^* of the dual projective space $(\mathbb{R}P^n)^*$. This duality is completely different from the usual projective duality and is similar to linear duality used in convex analysis. For the sake of completeness, we describe here this parallelism.

Projective duality. Projective space $\mathbb{R}P^n$ is the quotient of the linear space $\mathbb{R}^{n+1} \setminus 0$ by the proportionality relation. The dual projective space is, by definition, the quotient of the set of all non-zero covectors $\alpha \in (\mathbb{R}^{n+1})^* \setminus 0$ by the proportionality relation.

There is a one-to-one correspondence between the hyperplanes of the space and the points of the dual space. More generally, to any subspace $L \subset \mathbb{R}P^n$, there corresponds the dual subspace $L^* \subset (\mathbb{R}P^n)^*$ of all hyperplanes containing L, and the reflexivity property $(L^*)^* = L$ holds.

For any set $A \in \mathbb{R}P^n$, we define its dual set $A_p^* \subset (\mathbb{R}P^n)^*$ to be the set of all hyperplanes in $\mathbb{R}P^n$ disjoint from A. (The symbol A^* denotes the dual space, so we introduce the new notation A_p^*.)

Proposition. 1. *If A is non-empty, then the set A_p^* is contained in some affine chart of the dual space.*

2. *The set A_p^* is projectively separable.*

Proof. 1. If A is non-empty, then it contains a point b. The hyperplane $b^* \in (\mathbb{R}P^n)^*$ corresponding to the point b does not intersect A_p^*. Therefore, the set A_p^* is contained in the affine chart $(\mathbb{R}P^n)^* \setminus b^*$.

2. If a hyperplane $L \subset \mathbb{R}P^n$ considered as a point in the space $(\mathbb{R}P^n)^*$ does not belong to the set A_p^*, then, by definition, the hyperplane L intersects the set A. Let $b \in A \cap L$. The hyperplane b^* dual to the point b does not intersect the set A_p^*. So this hyperplane separates the point corresponding to the hyperplane L from the set A_p^*. □

The following theorem gives a full description of the set $(A_p^*)_p^*$.

Theorem. *For any set $A \subset \mathbb{R}P^n$, the corresponding set $(A_p^*)_p^*$ consists of all points a such that any hyperplane containing a intersects the set A. In other words, the set $(A_p^*)_p^*$ coincides with the projectively separable hull of the set A.*

Proof. A point a belongs to $(A_p^*)_p^*$ if and only if the corresponding hyperplane $a^* \subset (\mathbb{R}P^n)^*$ is disjoint from the set A_p^*. To any point $p \in (\mathbb{R}P^n)^*$ of this hyperplane, there corresponds a hyperplane $p^* \subset \mathbb{R}P^n$ containing the point a. The point $p \in (\mathbb{R}P^n)^*$ does not belong to A_p^* if and only if the hyperplane $p^* \subset \mathbb{R}P^n$ intersects the set A. So the condition that all points of the hyperplane $a \subset (\mathbb{R}P^n)^*$ do not belong to A_p^* means that all hyperplanes in $\mathbb{R}P^n$ containing the point a intersect the set A. □

Corollary. *The reflexivity property $(A_p^*)_p^* = A$ holds for all projectively separable subsets of projective space and only for them.*

Linear duality. The definition of affine separability differs from that of projective separability: the former involves closed half-spaces, while the latter involves hyperplanes. We can do the same with the duality theory developed above and define the set A_a^* corresponding to a subset A of an affine space as the set of all closed half-spaces disjoint from A. This definition is not very convenient, because the set of all closed half-spaces does not have the structure of affine space. Moreover, this set is topologically different from affine space: it is homeomorphic to the sphere S^n

with two points removed (one point corresponding to the empty set and the other to the whole space). We can overcome this difficulty by considering instead the set of all closed half-spaces with some fixed point removed and one element added (this element corresponds to the empty set regarded as a half-space at infinite distance from the fixed point). This set has a natural structure of affine space. Namely, taking the fixed point as the origin and denoting the obtained linear space by $\mathbb{R}^n$, we can parameterize the set described above by $(\mathbb{R}^n)^*$: to each nonzero $\alpha \in (\mathbb{R}^n)^*$ we assigns the closed half-space defined by inequality $\langle \alpha, x\rangle \geq 1$. To $\alpha = 0$ the empty set (defined by the same inequality $\langle \alpha, x\rangle \geq 1$) is assigned.

When dealing with affine duality, it is more convenient to consider only sets containing some fixed point. Taking this point as the origin, we get the well-known theory of affine duality, which is parallel to the theory of projective duality. Its main points are as follows.

To any subset A of linear space $\mathbb{R}^n$, there corresponds the subset A_l^* of the dual space $(\mathbb{R}^n)^*$ consisting of all $\alpha \in (\mathbb{R}^n)^*$ such that the inequality $\langle \alpha, x\rangle < 1$ holds for all $x \in A$.

Proposition. *For any set $A \subset \mathbb{R}^n$ containing the origin, the corresponding dual set A_l^* in the dual space has the property of affine separability. In particular, it is convex.*

Proposition. *For any set $A \subset \mathbb{R}^n$ containing the origin, the set $(A_l^*)_l^*$ consists of all points $a \in \mathbb{R}^n$ such that any closed half-space containing a intersects the set A. In other words, the set $(A_l^*)_l^*$ is equal to the affinely separable hull of the set A.*

Corollary. *The reflexivity property $(A_p^*)_p^* = A$ holds for all affinely separable convex sets containing the origin and only for them.*

3. L-Duality

In this section, we define an L-duality. A subset A of projective space $\mathbb{R}P^n$ disjoint from some subspace L, is L-dual to a subset $A_L^\perp$ of the dual projective space $(\mathbb{R}P^n)^*$ disjoint from the subspace L^*.

Any subset C in the projective space $(\mathbb{R}P^n)^*$ can be considered as a subset of the set of all hyperplanes in the projective space $\mathbb{R}P^n$. We shall also denote it by C.

Let L be some projective subspace of $\mathbb{R}P^n$, and let A be an arbitrary set disjoint from L. For a hyperplane π not containing the subspace L, we denote the subspace $L \cap \pi$ by L_π. Consider quotient space $(\mathbb{R}P^n)/L_\pi$. The image π_L of the hyperplane π is a hyperplane in the quotient space $(\mathbb{R}P^n)/L_\pi$.

Definition. We say that a hyperplane π *belongs to the L-dual set $A_L^\perp$* if π does not contain L and the hyperplane π_L is contained in the projection of the set A on the quotient space $(\mathbb{R}P^n)/L_\pi$.

In other words, a hyperplane π belongs to the set $A_L^\perp$ if the projection of π from the center L_π belongs to B_p^*, where B is the complement to the projection of the set A on the space $\mathbb{R}P^n/L_\pi$.

Another description of the set $A_L^\perp$ is as follows. The complement $\mathbb{R}P^n \setminus L$ to the subspace L is fibered by the spaces $N \supset L$ of dimension $\dim N = \dim L + 1$.

A hyperplane π belongs to $A_L^{\perp}$ if and only if the intersection of any fiber N with the set $A \cap \pi$ is non-empty, i. e., $N \cap A \cap \pi \neq \varnothing$. In other words, $\pi \in A_L^{\perp}$ if and only if π intersects the section of A by any $(\dim L + 1)$-dimensional space containing L.

Example. Let L be a hyperplane, and let A be a set disjoint from L, i. e., $A \cap L = \varnothing$. Then $A_L^{\perp}$ is the union of all hyperplanes intersecting the set A. In other words, the set $A_L^{\perp}$ is the complement to the set A_p^*. Indeed, in this case, the only space N containing L is projective space $\mathbb{R}P^n$ itself. Note that, in this case, the set L-dual to A does not depend on the choice of the hyperplane L (we can take any L disjoint from A).

Proposition. *If $A \subset B$ and $B \cap L = \varnothing$, then $(A_L^{\perp}) \subset (B_L^{\perp})$.*

Proof. If a hyperplane intersects all the sections $A \cap N$, then it intersects all the sections $B \cap N$. □

Proposition. *Let M be a projective subspace in $\mathbb{R}P^n$ not intersecting L and having maximal possible dimension, i. e., such that $\dim M = \dim L^* = n - \dim L - 1$. Then $M_L^{\perp} = M^*$.*

Proof. Any section of M by a $(\dim L+1)$-dimensional space containing L is a point, and any point of M is a section of M by such a space. By the definition of $M_L^{\perp}$, a hyperplane π belongs to $M_L^{\perp}$ if and only if it intersects all such sections, i. e., contains all points of M. This is exactly the definition of M^*. □

Let $L^* \subset (\mathbb{R}P^n)^*$ be the space dual to L. What can be said about (i) the sections of the set $A_L^{\perp}$ by $(\dim L^* + 1)$-dimensional spaces $N \supset L^*$ and (ii) the projections of the set $A_L^{\perp}$ from the $(\dim L^* - 1)$-dimensional subspaces T of the space L^*? We give answers to these questions below.

Sections of L-dual sets. First, recall of the duality between sections and projections. Let N be a projective subspace in $(\mathbb{R}P^n)^*$. Consider the subspace $N^* \subset \mathbb{R}P^n$ dual to N. We will need the isomorphism and the projection described below.

There is a natural isomorphism between the space dual to the quotient space $\mathbb{R}P^n/N^*$ and the space N. This isomorphism is the projectivisation of the natural isomorphism between the space dual to the quotient space and the subspace of the dual space dual to the kernel of the factorisation. Each hyperplane containing the space N^* projects onto a hyperplane in $\mathbb{R}P^n/N^*$. (If a hyperplane does not contain the space N^*, then its projection is the whole space $\mathbb{R}P^n/N^*$.)

Using this isomorphism, we can describe the section of a set $C \subset (\mathbb{R}P^n)^*$ by the space N in terms of space $\mathbb{R}P^n$. Consider the subset $C_{N^*} \subset (\mathbb{R}P^n)^*$ of the set of hyperplanes C which consists of all hyperplanes containing N^* (equivalently, $C_{N^*} = C \cap N$). Each hyperplane from C_{N^*} projects onto a hyperplane in the quotient space $\mathbb{R}P^n/N^*$. But the space $(\mathbb{R}P^n/N^*)^*$ is identified with the space N. Thus, the required section $C \cap N$ is obtained from the set C_{N^*} by projection and identification.

Theorem 1. *Let A be a subset of $\mathbb{R}P^n$ not intersecting L, and let N be any subspace of $(\mathbb{R}P^n)^*$ containing L^* as a hyperplane (i. e., such that $\dim N = \dim L + 1$ and*

$N \supset L^*$). Then the section $A_L^\perp \cap N$ is equal to B_p^*, where $B \subset (\mathbb{R}P^n/N^*)$ is the complement to the projection of the set A on the space $(\mathbb{R}P^n)/N^*$.

Proof. This theorem follows from the description of the sections of the subsets of $(\mathbb{R}P^n)^*$ given above. Consider the set of hyperplanes $C = A_L^\perp$. By the definition of the set $A_L^\perp$, the set C_{N^*} consists of all hyperplanes containing the projective space N^* and such that, projecting them from N^* onto $\mathbb{R}P^N/N^*$, we obtain subsets of A. In other words, their projections are hyperplanes in $\mathbb{R}P^N/N^*$ not intersecting the complement to the projection of the set A. *Vice versa*, any hyperplane not intersecting this complement B is, by the definition of the set $A_L^\perp$, a projection of some hyperplane belonging to the set C_{N^*}. Therefore, $A_L^\perp \cap N = B_p^*$. □

Projections of L-dual sets. Recall of the duality between projections and sections.

Let Q denote the subspace in $\mathbb{R}P^n$ dual to the center of projection $T \subset (\mathbb{R}P^n)^*$. There is a natural isomorphism between the space Q^*, which consists of all hyperplanes of the space Q, and the quotient space $(\mathbb{R}P^n)^*/T$. Namely, consider the points of $(\mathbb{R}P^n)^*/T$ as equivalence classes in the set of all hyperplanes in the space $\mathbb{R}P^n$ not containing the space Q with respect to the following equivalence relation: two hyperplanes are equivalent if and only if their intersections with Q coincide. This intersection is the hyperplane in the space Q that corresponds to the equivalence class under consideration.

The projection of a subset C of $(\mathbb{R}P^n)^*$ from a center T can be described as follows. A set of hyperplanes C in $\mathbb{R}P^n$ determines some set of hyperplanes $C(Q)$ in the subspace $Q = T^*$; namely, a hyperplane $Q_1 \subset Q$ belongs to the set $C(Q)$ if and only if there exists a hyperplane belonging to the set C and intersecting Q exactly in Q_1. The projection of the set C from the center T is exactly the set $C(Q)$ of hyperplanes in Q after Q^* and $(\mathbb{R}P^n)^*/T$ are identified.

Theorem 2. *Let A be a set in $\mathbb{R}P^n$ not intersecting L, and let T be a hyperplane in the dual space $L^* \subset (\mathbb{R}P^n)^*$. Then the projection of the set $A_L^\perp$ from the center T can be described as the set of all hyperplanes p in the space $Q = T^* \supset L$ such that, for each of them, there exists a hyperplane $\pi \subset A_L^\perp$ whose intersection with Q is p, i. e., $p = \pi \cap Q$.*

Proof. This theorem follows from the description of the projections of subsets $C \subset (\mathbb{R}P^n)^*$ given above. □

Definition. We say that a set A is *coseparable* relative to L if $A \cap L = \varnothing$ and, for any hyperplane $L_1 \subset L$, the complement to the projection of the set A from the center L_1 has the property of affine separability in the space $(\mathbb{R}P^n)/L_1$.

Corollary. *If all conditions of Theorem 2 and the set A is coseparable relative to L, then the complement to the projection of the set $A_L^\perp$ from the center T is dual to the section $A \cap T^*$ (i. e., it is equal to $(A \cap T^*)_p^*$).*

Description of the set $(A_L^\perp)_{L^*}^\perp$. Let A be a subset of $\mathbb{R}P^n$ not intersecting L, and let L^* be the subspace of $(\mathbb{R}P^n)^*$ dual to L. What can be said about the subset of $\mathbb{R}P^n$ L^*-dual to the subset $A_L^\perp$ of the space $(\mathbb{R}P^n)^*$? From Theorems 1 and 2, we easily obtain the folowing description of this set $(A_L^\perp)_{L^*}^\perp$.

Theorem 3. *The set* $(A^{\perp}_L)^{\perp}_{L^*}$ *does not intersect* L *and consists of all points* $a \in \mathbb{R}P^n$ *satisfying the following condition. Let* L_a *be the space spanned by* L *and* a*. For any hyperplane* p *in* L_a *containing the point* a *(i. e., such that* $a \in p \subset L_a$*), there is a hyperplane* $\pi \subset \mathbb{R}P^n$ *such that* $\pi \in A^{\perp}_L$ *and* $p = \pi \cap L_a$*.*

Proof. According to Theorem 1 (applied to the subset $A^{\perp}_L$ of the space $(\mathbb{R}P^n)^*$ and to the subspace L^* of this space), the section of the set $(A^{\perp}_L)^{\perp}_{L^*}$ by the subspace L_a can be described as the set of hyperplanes in $(\mathbb{R}P^n)^*/L^*_a$ disjoint from the complement to the projection of the set $A^{\perp}_L$ on the space $(\mathbb{R}P^n)^*/L^*_a$.

So the point $a \in \mathbb{R}P^n$ lies in $(A^{\perp}_L)^{\perp}_{L^*}$ if and only if the hyperplane in $(\mathbb{R}P^n)^*/L^*_a$ corresponding to this point $a \in \mathbb{R}P^n$ $(a \in L_a)$ is contained in the projection of the set $A^{\perp}_L$. This means that any hyperplane p of L_a $(P \subset L_a)$ containing the point a lies in the projection of the set $A^{\perp}_L$ when considered as a point of the space $(\mathbb{R}P^n)^*/L^*_a$. According to Theorem 2, this means that, for the hyperplane p, there exists a hyperplane $\pi \in A^{\perp}_l$ such that $\pi \cap L_a = p$. □

Let us reformulate Theorem 3. A point a belongs to the set $(A^{\perp}_L)^{\perp}_{L^*}$ if the following two conditions hold.

Condition 1. The point a in the space L_a spanned by L and a has the following property: any hyperplane $p \subset L_a$ containing a intersects the set $L_a \cap A$. In other words, the point a belongs to the set $((L_a \cap A)^*_p)^*_p$.

Condition 2. The projection of the point a from any center $L_1 \subset L$, where L_1 is a hyperplane in L, belongs to some hyperplane in the space $(\mathbb{R}P^n)/L_1$ contained in the projection of the set A on the space $(\mathbb{R}P^n)/L_1$.

Theorem 4. *Conditions* 1 *and* 2 *are equivalent to the condition that the point* a *belongs to the set* $(A^{\perp}_L)^{\perp}_{L^*}$*.*

Proof. Indeed, according to Theorem 3, if $a \in (A^{\perp}_L)^{\perp}_{L^*}$, then any hyperplane p in L_a containing the point a is the intersection of L_a and a hyperplane $\pi \in A^{\perp}_L$. This means that, first, the hyperplane p intersects A and, secondly, that the projection of the point a from $L_1 = L \cap \pi$ belongs to in a hyperplane in the factor-space $\mathbb{R}P^n/L_1$, which, in turn, is contained in the projection of the set A. The first property is equivalent to Condition 1, and the second is equivalent to Condition 2. □

Corollary. *Suppose that a set* A *does not intersect the space* L *and the intersection of* A *with any subspace* N *containing* L *as a hyperplane is projectively the separable in projective space* N*. Then* $(A^{\perp}_L)^{\perp}_{L^*} \subset A$*.*

Proof. Indeed, condition 1 guarantees that, for any space N containing L as a hyperplane, the inclusion $(A^{\perp}_L)^{\perp}_{L^*} \cap N \subset ((N \cap A)^*_p)^*_p$ holds. But $((N \cap A)^*_p)^*_p = N \cap A$, since $N \cap A$ is projectively separable. Therefore, $(A^{\perp}_L)^{\perp}_{L^*} \subset A$. □

Corollary. *Suppose that a set* A *is coseparable relative to* L*. Then the intersection of the set* $(A^{\perp}_L)^{\perp}_{L^*}$ *with any space* N *containing* L *as a hyperplane depends only on the subset* $A \cap N$ *of the projective space* N *and coincides with the set* $((A \cap N)^*_p)^*_p$*. In particular,* $A \subseteq (A^{\perp}_L)^{\perp}_{L^*}$*.*

Proof. If the set A is coseparable relative to L, then condition 2 holds for all points satisfying condition 1. This is exactly what the corollary asserts. □

Properties of L-coseparable and L-separable sets. Let us summarize the facts about L-coseparable and L-separable subsets of projective space proved above.

Let a subset A of projective space $\mathbb{R}P^n$ be coseparable relative to a space L; suppose that any section of A by a space containing L as a hyperplane is projectively separable.

Then the set $A_L^\perp$ in the dual projective space $(\mathbb{R}P^n)^*$ has the same properties relative to the dual space L^*. Moreover, any section of $A_L^\perp$ by a subspace N containing L^* as a hyperplane is dual to the set B (i. e., is equal to B_p^*), where B is the complement to the projection of the set A on $(\mathbb{R}P^n)/N^*$ from the center N^*. The projection of the set $A_L^\perp$ from the center T, where T is an arbitrary hyperplane in space L^*, is dual to the section of A by T^* (i. e., it is equal to $(A \cap T^*)_p^*$). Furthermore, the reflexivity relation $(A_L^\perp)_L^\perp = A$ holds.

If the set $A_L^\perp$ contains a projective space M^* of dimension equal to the dimension of the space L, then the set A contains its dual space M of dimension equal to the dimension of the space L^*.

L-convex-concave sets are L-separable and L-coseparable, because closed sets and open sets are both separable. Therefore, L-convex-concave sets have all the aforementioned properties.

Semialgebraic L-convex-concave sets. Here, we shall apply integration along the Euler characteristic introduced by Viro (see [Vi]). We denote the Euler characteristic of a set X by $\chi(X)$.

Theorem. *Let A be an L-convex-concave closed semialgebraic set in $\mathbb{R}P^n$, and let $\dim L = k$. Then, for any hyperplane $\pi \subset \mathbb{R}P^n$, $\chi(A \cap \pi)$ is equal to $\chi(\mathbb{R}P^{n-k-1})$ or to $\chi(\mathbb{R}P^{n-k-2})$. In the first case, the hyperplane π considered as a point of $(\mathbb{R}P^n)^*$ belongs to the set $A_L^\perp$ L-dual to A. In the second case, the hyperplane π does not belong to the set $A_L^\perp$.*

Proof. The complement to L in $\mathbb{R}P^n$ is a union of non-intersecting fibers, each fiber being a $(k+1)$-dimensional space N containing L. The set A is L-convex-concave, so its intersection with each fiber N is convex and closed. Therefore, for each space N, the intersection $A \cap N \cap \pi$ of the set $A \cap N$ with a hyperplane π either is empty or is a closed convex set.

Suppose that the hyperplane π does not contain the space L. Let us denote the space $L \cap \pi$ by L_π. In the quotient space $\mathbb{R}P^n/L_\pi$, we have a fixed point $\pi(L)$ (the projection of the space L), a set B (the complement to the projection of the set A from L_π), and a hyperplane π_L (the projection of the hyperplane π). To each point a of the hyperplane π_L in the quotient space, there corresponds a space $N(a)$ in $\mathbb{R}P^n$ ($N(a) \supset L$) whose projection is equal to the line passing through a and $\pi(L)$. The intersection $N(a) \cap A \cap \pi$ is empty if a belongs to the set B. Otherwise, the intersection $N(a) \cap A \cap \pi$ is a closed convex set. The Euler characteristic of the set $N(a) \cap A \cap \pi$ is equal to 0 in the first case, and it is equal to 1 in the second case. Using the Fubini theorem for an integral along the Euler characteristics for

FIGURE 1. (a) A cone K; (b) A set A pointed with respect to the cone K.

the projection of the set $A \cap \pi$ on the quotient space $\mathbb{R}P^n/L_\pi$, we obtain

$$\chi(A \cap \pi) = \chi(\pi_L \setminus (\pi_L \cap B)).$$

So $\chi(A \cap \pi) = \chi(\pi_L) = \chi(\mathbb{R}P^{n-k-1})$, if $\pi_L \cap B = \varnothing$. Otherwise, i. e., if $\pi_L \cap B \neq \varnothing$, we have $\chi(A \cap \pi) = \chi(\mathbb{R}P^{n-k-2})$. In the first case, $\pi_L \in A_L^\perp$ by definition, and in the second case, $\pi_L \notin A_L^\perp$. This proves the theorem for the hyperplanes not containing the space L. If $L \subset \pi$, similar considerations imply $\chi(\pi \cap A) = \chi(\mathbb{R}P^{n-k-2})$. □

Corollary. *For a semi-algebraic L-convex-concave set A, the L-dual set is defined canonically* (*i. e., $A_L^\perp$ does not depend on the choice of the space L relative to which the set A is L-convex-concave*).

Remark. For the semialgebraic L-convex-concave sets, the reflexivity relation

$$(A_L^\perp)_{L^*}^\perp = A$$

can be proved by using only this theorem and the Radon transform of the integral along the Euler characteristic (see [Vi], [PKh]).

4. Duality between Pointed Convex Sections of Convex-Concave Sets and an Affine Dependence of Convex Sections on a Parameter

In this section, we define the properties of being pointed (with respect to a cone) and of affine dependence on a parameter (for parameters belonging to some convex domain) of sections.

We begin with the affine versions of these notions and then give the corresponding projective definitions. We prove that the property of being pointed is dual to an affine dependence on a parameter.

Pointed sections. We start with affine settings. Let K be a pointed (i. e., containing no linear subspaces) closed convex cone in a linear space N with vertex at the origin.

We say that a set A is *pointed* with respect to K if there is a point $a \in A$ such that the set A lies entirely in the shifted cone $(K + a)$ with vertex at the point a. This point a is called the *vertex* of the set A relative to the cone K. Obviously, the vertex of the set A relative to K is determined uniquely.

In affine space, we deal with pointed cones K which are unions of rays beginning at the vertex of the cone containing no lines.

FIGURE 2. (a) A projectively pointed cone $\tilde{K}$; (b) A set A pointed with respect to the cone $\tilde{K}$.

In the projective setting, it is more natural to consider cones $\tilde{K}$ which are unions of lines. Such a cone $\tilde{K}$ is said to be *projectively pointed* if the set of lines lying in the cone forms a convex set in $\mathbb{R}P^{n-1}$. Evidently, a cone $\tilde{K}$ is projectively pointed if and only if it is a union of an affine pointed cone K with its opposite cone $(-K)$: $\tilde{K} = K \cup (-K)$.

We say that a set A in affine space is *pointed with respect to a cone* $\tilde{K} = K \cup (-K)$ if the set A is pointed with respect to both the cone K and the cone $(-K)$.

A set A pointed with respect to a cone $\tilde{K}$ has two vertices a and b relative to the cones K and $(-K)$, respectively.

The following statement is evident.

Proposition. *Suppose that a connected set A is pointed with respect to a cone $\tilde{K} = K \cup (-K)$ and a and b are the vertices of A relative to $\tilde{K}$. Let $\tilde{Q}$ be a hyperplane intersecting $\tilde{K}$ at only one point (the origin). Then an affine hyperplane Q parallel to the hyperplane $\tilde{Q}$ intersects the set A if and only if Q intersects the segment joining the points a and b. Vice versa, if a connected set A with fixed points a and b has this property, then the set A is pointed with respect to the cone $\tilde{K} = K \cup (-K)$ and a and b are the vertices of A.*

Now, consider the projective setting. Suppose that N is a projective space, $L \subset N$ is a fixed hyperplane, and $\Delta \subset L$ is a closed convex set in L.

We say that a connected set $A \subset N$ not intersecting the hyperplane L is *pointed with respect to the convex set* Δ if there exist two points a and b in the set A (the *vertices* of the set A with respect to Δ) such that any hyperplane p in N disjoint from the convex set $\Delta \subset L$ intersects A if and only if p intersects the segment joining the points a and b and lying in the affine space $N \subset L$.

This projective definition is a projective reformulation of the affine definition. Indeed, projective space is a linear space to which a hyperplane at infinity is added. To a convex set Δ lying in the hyperplane at infinity, there corresponds a pointed cone $\tilde{K}$ being the union of all lines passing through the origin and points of the set Δ.

According to the proposition, a set A in the affine space $N \setminus L$ is pointed with respect to the cone $\tilde{K}$ if and only if this set A considered as a subset of projective space is pointed with respect to the convex set $\Delta = \tilde{K} \cap L$.

Families of convex sets affinely dependent on parameters. We begin with an affine setting. Fix a linear subspace N of linear space $\mathbb{R}^n$. Linear space $\mathbb{R}^n$ is fibered by affine subspaces N_m parallel to N and parameterized by the points m of the quotient space $\mathbb{R}^n/N$. Fix a convex domain Δ in the space of parameters $\mathbb{R}^n/N$. Suppose that, for each point $m \in \Delta$, a closed convex set $A_m \subset N_m$ is given.

We say that a *family of convex sets* $\{A_m\}$ *depends affinely on the parameter* $m \in \Delta$ if, for any two points $m_1, m_2 \in \Delta$ and any $0 \leq t \leq 1$, the set A_{m_t} corresponding to the parameter $m_t = tm_1 + (1-t)m_2$ is a linear combination $tA_{m_1} + (1-t)A_{m_2}$ of sets A_{m_1} and A_{m_2} in the sense of Minkowski.

Proposition. *A family of convex sets A_m, where $m \in \Delta$, depends affinely on the parameter if and only if for any simplex $\Delta(a_1, \ldots, a_k) \subset \Delta$ with linearly independent vertices $a_1, \ldots, a_k \in \Delta$, the convex hull of the union of the sets $A_{a_1} \ldots, A_{a_k}$ coincides with the union of the sets A_m over all parameters $m \in \Delta(a_1, \ldots, a_k)$.*

The particular case of one-dimensional space $N = (l)$ is especially simple. In this case, the convex sets A_m are simply segments, and the proposition reads as follows.

Proposition. *A family of parallel segments A_m in $\mathbb{R}^n$ depends affinely on the parameter m belonging to a convex domain $\Delta \subset \mathbb{R}^n/(l)$ if and only if there exist two hyperplanes Γ_1 and Γ_2 in the space $\mathbb{R}^n$ such that, first, for any $m \in \Delta$ the end points of the segment $A(m)$ coincide with intersection points of the line N_m with the hyperplanes Γ_1 and Γ_2 and, secondly, the projection along N of the intersection of Γ_1 and Γ_2 is disjoint from the interior of Δ.*

The general definition of affine dependence on a parameter can be reduced to the case of one-dimensional space by using projections. Let Q be a subspace of the space N. The quotient space $\mathbb{R}^n/Q$ contains the subspace $\pi(N) = N/Q$. The spaces $(\mathbb{R}^n/Q)/\pi(N)$ and $\mathbb{R}^n/N$ are naturally isomorphic; we shall use this isomorphism below.

We say that *a family of convex sets $A_m \subset N_m$ depends affinely on the parameter $m \in \Delta \subset \mathbb{R}^n/N$ in the direction of the hyperplane Q in the space N* if the segments $\pi(A_m)$ on the lines N_m/Q, where $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/Q$ is the projection, depend affinely on the parameter $m \in \Delta$. (Using the isomorphism of $(\mathbb{R}^n/Q)/\pi(N)$ and $\mathbb{R}^n/N$, we consider $\Delta \subset \mathbb{R}^n/N$ as a set in $(\mathbb{R}^n/G)/\pi(N)$.)

Theorem. *A family of convex sets $A_m \subset N_m$ depends affinely on the parameter $m \in \Delta$ if and only if the family $A_m \subset N_m$ depends affinely on the parameter $m \in \Delta$ in the direction of Q for any hyperplane Q.*

Proof. Taking a subspace M transversal to N, we identify all parallel spaces N_m (two points of different sections are identified if they lie in the same translate of M). Then all dual spaces N_m^* are identified with the space N^*, and all support functions $H_m(\xi) = \max\limits_{x \in A_m} (\xi, x)$ of convex sets A_m can be considered as functions on the same space N^*.

To a linear combination (in Minkowski sense) of convex sets, there corresponds a linear combination of their support functions. So the dependence of the family of convex sets A_m on the parameter $m \in \Delta$ is affine if and only if, for any fixed covector $\xi \in N^*$, the support function $H_m(\xi)$ is linear with respect to the parameter m.

Let us rewrite this condition for ξ and $-\xi$ simultaneously. We denote the hyperplane in N defined by the equation $(\xi, x) = (-\xi, x) = 0$ by Q. Consider the projection of the set $A = \bigcup_{m \in \Delta} A_m$ along the space Q. The projection $\pi(A)$ lies in the space $\mathbb{R}^n/Q$ with marked one-dimensional subspace $l = N/Q$. Each line l_m, where $m \in \mathbb{R}^n/N = (\mathbb{R}^n/Q)/l$, contains a segment $\pi(A_m)$ equal to the projection of the convex set A_m.

By assumption, the segments $\pi(A_m)$ lie between two hyperplanes Γ_1 and Γ_2. Furthermore, the endpoints $x(m)$ and $y(m)$ of these segments lie on the line l_m, and they are defined by equations $H_m(\xi) = \langle \xi, x(m) \rangle$ and $H_m(-\xi) = \langle -\xi, y(m) \rangle$. Therefore, the affine dependence of the convex sets A_m, $m \in Q$, in the direction Q means that the support functions $H_\xi(m)$ and $H_{-\xi}(m)$, where ξ are covectors orthogonal to Q, are first-degree polynomials in $m \in \Delta$. Since this is true for any hyperplane $Q \subset N$, the function $H_\xi(m)$ depends linearly on m for any fixed ξ. $\square$

Now, consider the projective settings. Instead of linear space $\mathbb{R}^n$ fibered by affine subspaces N_m parallel to a space N and parameterized by the points of the quotient space $\mathbb{R}^n/N$, we shall deal with a projective space $\mathbb{R}P^n$ containing a projective subspace L and fibered by the subspaces N_m of dimension $\dim N_m = \dim L + 1$ which contain the space L. The subspaces N_m are parameterized by the points of the quotient space $M = (\mathbb{R}P^n)/N$. Consider the parameters m belonging to a convex set $\Delta \subset M$.

Let $T \subset L$ be a hyperplane in L. We denote a projection of the projective space from the center T by π. The projection of the space L is a point $\pi(L)$. The projection of the space N is a line l belonging to the bundle of all lines $l_m = \pi(N_m)$ containing the marked point $\pi(L)$. After the natural identification of the quotient spaces $(\mathbb{R}P^n)/L$ and $(\mathbb{R}P^n/T)/\pi(L)$, the space $N_m \subset \mathbb{R}P^n$ and the line $l_m = \pi(N_m) \subset \mathbb{R}P^n/T$ correspond to the same parameter $m \in \mathbb{R}P^n/L = (\mathbb{R}P^n/T)/\pi(L)$. The domain $\Delta \subset \mathbb{R}P^n/L$ can be considered as a domain in the space $(\mathbb{R}P^n/T)/\pi(L)$.

We use the following notation. Let Γ_1 and Γ_2 be two hyperplanes in projective space not containing the point $\pi(L)$, and let l be a line containing this point. Intersection points of Γ_1 and Γ_2 with the line l divide this line into two segments. The segment not containing the point $\pi(L)$ is called the segment between hyperplanes Γ_1 and Γ_2 on the line l exterior relative to the point $\pi(L)$.

Let A be a set not intersecting space L and such that its sections A_m by the spaces $N_m \supset L$ are convex. We say that *the sections A_m depend affinely on the parameter m* belonging to a convex domain $\Delta \subset \mathbb{R}P^n/L$ *in the direction of the hyperplane* $T \subset L$ if the sections of the set $\pi(A)$ by the lines l_m containing the point $\pi(L)$ depend affinely on $m \in \Delta \subset \mathbb{R}P^n/L$ $(= (\mathbb{R}P^n/T)/(\pi(L))$. In other words, there exist two hyperplanes Γ_1 and Γ_2 in $\mathbb{R}P^n/T$ not containing $\pi(L)$ and such that, first, the intersection of $\pi(A)$ with any line l_m, $m \in \Delta$, is equal to the segment of the line l_m lying between Γ_1 and Γ_2 exterior relative to $\pi(L)$, and, secondly, the projection of $\Gamma_1 \cap \Gamma_2$ on $\mathbb{R}P^n/L$ does not intersect Δ.

Now, we can give a definition of affine dependence of sections on a parameter belonging to a convex domain in the space of parameters.

We say that *the sections A_m of the set A* by projective spaces $N_m \supset L$ *depend affinely on the parameter* m in a domain Δ if A_m depend affinely on the parameter m in the domain Δ with respect to any hyperplane $T \subset L$. The following statement can be easily checked.

Proposition. *Let Γ be a projective hyperplane containing the space L and such that its projection to the space $(\mathbb{R}P^n)/L$ does not intersect a convex set $\Delta \subset \mathbb{R}P^n/L$. Consider the affine chart $U = \mathbb{R}P^n \setminus \Gamma$ of the projective space. The sections of a set $A \subset \mathbb{R}P^n$, where $A \cap L = \varnothing$, by spaces N_m depend affinely on the parameter m in the domain Δ if and only if the sections of the set $A \cap U$ in the affine space U by the parallel spaces $N_m \setminus \Gamma$ depend affinely on the parameter m in the domain $\Delta \subset ((\mathbb{R}P^n)/L) \setminus (\Gamma/L)$.*

Duality. Let Δ be a convex domain in the space L, and let Δ_p^* be the dual convex domain in the space $(\mathbb{R}P^n)^*/L^*$. The space $(\mathbb{R}P^n)^*/L^*$ parameterizes the $(\dim L^* + 1)$-dimensional subspaces of $(\mathbb{R}P^n)^*$ containing L^*. The domain Δ^* corresponds to subspaces $Q^* \subset (\mathbb{R}P^n)^*$ of this type which are dual to subspaces $Q \subset L$ not intersecting the domain Δ.

Theorem. *Let A be an L-convex-concave subset of projective space $\mathbb{R}P^n$. A section $A \cap N$ of the set A by a $(\dim L + 1)$-dimensional subspace N containing L is pointed relative to a convex domain $\Delta \subset L$ if and only if the following dual condition holds: the subset $A_L^{\perp}$ of the dual space $(\mathbb{R}P^n)^*$ L-dual to the A depends affinely on the parameter belonging to the domain Δ_p^* in the direction of the hyperplane $N^* \subset L^*$.*

Proof. The set A is L-convex-concave, so the section $A \cap N$ is dual to the complement to the projection from the center $N^* \subset L^*$ of the set $A_L^{\perp}$.

Let a and b be the vertices of the pointed set $A \cap N$ relative to the convex set $\Delta \subset L$. Fix a hyperplane q_L in L disjoint from the convex set $\Delta \subset L$. Consider a one-dimensional bundle $\{p^t\}$ of hyperplanes containing the space q_L in N. This bundle contains the following three hyperplanes: the hyperplane L, a hyperplane p_a containing the vertex a of the set A, and a hyperplane p_b containing the vertex b of the set A.

take the segment $[p_a, p_b]$ with endpoints p_a and p_b not containing the point L. Any hyperplane p^t (except the hyperplane L itself) intersects L in a subspace q_L, and q_L does not intersect Δ. The set A is pointed with respect to Δ, so a hyperplane p^{t_0} intersects $A \cap N$ if and only if the point p^{t_0} belongs to the segment $[p_a, p_b]$.

Consider the dual space $(\mathbb{R}P^n)^*$. To the section $A \cap N$ of the set A, there corresponds the projection of the set $A_L^{\perp}$ from the center N^*. Hyperplanes in N correspond to points in the quotient space $(\mathbb{R}P^n)^*/N^*$. In particular, the hyperplane L in N corresponds to the marked point $\pi(L^*)$ in the quotient space $(\mathbb{R}P^n)^*/N^*$, namely, the projection of the space L^* from the center N^*. The bundle of hyperplanes $\{p^t\}$ corresponds to a line passing through $\pi(L^*)$. This line intersects the projection of the set $A_L^{\perp}$ exactly in the segment $[p^a, p^b]$ not containing the point $\pi(L^*)$.

To different hyperplanes q_L in the space L, there correspond different one-dimensional bundles of hyperplanes $\{p^t\}$ in N, i. e., different lines in $(\mathbb{R}P^n)^*/N^*$, containing the marked point $\pi(L^*)$. The hyperplane q_L in the space L does not intersect Δ, so the dual space $q_L^* \supset L^*$ is parameterized by a point of $\Delta_p^* \subset (\mathbb{R}P^n)^*/L^*$. The projection of the space q_L^* from the center N^* is a line in the space $(\mathbb{R}P^n)^*/N$ parameterized by the same point of the domain Δ^*. Every such line intersects the projection of the set $A_L^\perp$ in the segment $[p^a, p^b]$. The point p^a lies in the hyperplane Γ^a of the space $(\mathbb{R}P^n)^*/N^*$ dual to the point $a \in N$. The point p^b lies in the hyperplane Γ^b of the space $(\mathbb{R}P^n)^*/N^*$ dual to the point $b \in N$.

The two hyperplanes Γ^a and Γ^b divide the space $(\mathbb{R}P^n)^*/N^*$ into two parts. Let $\Gamma(a, b)$ denote the closure of the part not containing the point $\pi(L^*)$. We have just proved that the set $\Gamma(a, b)$ and the projection of the set $A_L^\perp$ to the space $(\mathbb{R}P^n)^*/N^*$ have the same intersections with the lines passing through the point $\pi(L^*)$ and parameterized by points of the domain Δ_p^*. This completes the proof of the theorem. □

5. L-Convex-Concave Sets with Plane Sections Being Octagons with Four Pairs of Parallel Sides

Consider a subset A of $\mathbb{R}P^n$ convex-concave with respect to a one-dimensional space L. Fix four points $a_1, \dots, a_4$ lying on the line L in this order. These points divide L into four pairwise disjoint intervals $\langle a_1, a_2\rangle$, $\langle a_2, a_3\rangle$, $\langle a_3, a_4\rangle$, and $\langle a_4, a_1\rangle$. We denote their complements to L by $I_1 = [a_1, a_2] = L \setminus \langle a_1, a_2\rangle, \dots, I_4 = [a_4, a_1] = L \setminus \langle a_4, a_1\rangle$ (these segments are intersecting). In this section, we prove the main conjecture for L-convex-concave sets A whose sections $N \cap A$ by two-dimensional planes N containing the line L are pointed relative to the segments $I_1, \dots, I_4$.

Theorem. *Suppose that all plane sections $A \cap N$ of an L-convex-concave set $A \subset \mathbb{R}P^n$, where $\dim L = 1$, by two-dimensional planes N containing L are pointed with respect to four segments $I_1, \dots, I_4$ on the line L. Suppose also that the union of I_i coincides with L and that the complements $L \setminus I_i$ are pairwise disjoint. Then the set A contains a projective space M of dimension $(n - 2)$.*

Before proceeding to the proof we shall make two remarks.

First, the assumptions of the theorem about the convex-concave set A are easier to understand in an affine chart $\mathbb{R}^n$ not containing the line L. In this chart, the family of two-dimensional planes containing L becomes a family of parallel two-dimensional planes. In space $\mathbb{R}^n$, four classes of parallel lines are fixed, each passing through one of the points $a_1, \dots, a_4$ of the line L at infinity. The assumptions of the theorem mean that each section of the set A by a plane N is an octagon with sides belonging to these four fixed classes of parallel lines. (Some sides of this octagon may degenerate into a point, and the number of sides of the octagon $(A \cap N)$ will then be smaller than 8.)

Furthermore, there is a natural isomorphism between $(\mathbb{R}P^1)^*$ and $\mathbb{R}P^1$. Indeed, each point $c \in \mathbb{R}P^1$ of projective line is also a hyperplane in $\mathbb{R}P^1$. However, a segment $[a, b]$ on the projective line $\mathbb{R}P^1$ is dual to its *complement* $\langle a, b\rangle =$

$(\mathbb{R}P^1) \setminus [a, b]$, and it is not dual to itself. Indeed, by definition, the set Δ_p^* dual to a convex set Δ consists of all hyperplanes not intersecting Δ.

Proof of the theorem. Consider the dual projective space $(\mathbb{R}P^n)^*$ and its subspace L^*, $\dim L^* = (n-2)$, dual to the line L. The projective line $(\mathbb{R}P^n)^*/L^*$ isomorphic to the line dual to L is divided by the points $a_1^*, \ldots, a_4^*$ into four intervals $\langle a_1^*, a_2^*\rangle$, $\langle a_2^*, a_3^*\rangle$, $\langle a_3^*, a_4^*\rangle$, $\langle a_4^*, a_1^*\rangle$ dual to the segments $I_1, \ldots, I_4$. The set $A_L^\perp$ L-dual to A affinely depends on a parameter on these intervals, since the set A is pointed relative to the segments $I_1, \ldots, I_4$. Therefore, the set $A_L^\perp$ is a linear interpolation of its four sections. In other words, this set has four sections by the planes corresponding to $a_1^*, \ldots, a_4^*$, and all other sections of $A_L^\perp$ are affine combinations (in the sense of Minkowski) of the sections corresponding to the endpoints of the intervals. The L^*-convex-concave sets of this type contain a line (see the introduction and [KhN1]). Let us denote this line by l. The set A contains the $(n-2)$-dimensional space $l^* \subset \mathbb{R}P^n$ dual to the line l. □

6. Surgeries on Convex-Concave Sets

In this section, we describe two special surgeries on L-convex-concave subsets of $\mathbb{R}P^n$; one of them applies when $\dim L = n-2$ and the other one when $\dim L = 1$. These two surgeries are dual.

The first surgery: $\dim L = n - 2$**.** An $(n-2)$-dimensional subspace L of $\mathbb{R}P^n$ corresponds to a one-dimensional bundle of hyperplanes containing L. These hyperplanes are parameterized by the points of the projective line $\mathbb{R}P^n/L$. Fix two points a and b and a segment $[a, b]$ on this line being one of the two segments into which the points a and b divide the projective line $\mathbb{R}P^n/L$.

For any L-convex-concave set A and the segment $[a, b] \subset \mathbb{R}P^n/L$, we define a set $S_{[a,b]}(A)$, which is also L-convex-concave as follows. The hyperplanes Γ_a and Γ_b corresponding to the parameters a and b have the property $L = \Gamma_a \cap \Gamma_b$ and divide the set $\mathbb{R}P^n \setminus L$ into two half-spaces: the first half-space $\Gamma^1[a, b]$ is projected onto the segment $[a, b]$, and the second one $\Gamma^2[a, b]$ is projected onto its complement.

Let c be some point on the line $\mathbb{R}P^n/L$ not belonging to the segment $[a, b]$, and let Γ_c be the corresponding hyperplane in $\mathbb{R}P^n$.

Definition. The set $S_{[a,b]}(A)$ is defined by the following conditions:

(1) the set $S_{[a,b]}(A)$ does not intersect the space L, i. e., $S_{[a,b]}(A) \cap L = \varnothing$;
(2) the set $S_{[a,b]}(A) \cap \Gamma^1_{[a,b]}$ coincides with the convex hull of the union of the sections $A \cap \Gamma(a)$ and $A \cap \Gamma(b)$ in the affine chart $\mathbb{R}P^n \setminus \Gamma_c$;
(3) the set $S_{[a,b]}(A) \cap \Gamma^2_{[a,b]}$ coincides with $A \cap \Gamma^2_{[a,b]}$.

It is easy to see that the set $S_{[a,b]}(A)$ is well-defined, i. e., it does not depend on the choice of the hyperplane Γ_c.

Theorem. *For any L-convex-concave set A, the set $S_{[a,b]}(A)$ is also L-convex-concave.*

Proof. Any section of the set $S_{[a,b]}(A)$ by a hyperplane Γ_d containing L is convex. Indeed, if $d \notin [a, b]$, then $\Gamma_d \cap S_{[a,b]}(A) = \Gamma_d \cap A$, and the set $\Gamma_d \cap A$ is convex by

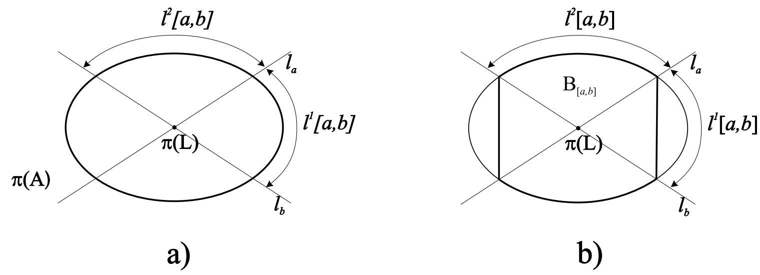

FIGURE 3. (a) The complement B to the projection $\pi(A)$ of the set A; (b) The complement $B_{[a,b]}$ to the projection $\pi(S_{[a,b]}(A))$ of the set $S_{[a,b]}(A)$.

definition. Otherwise, i. e., if $d \in [a, b]$, $\Gamma_d \cap S_{[a,b]}(A)$ is a linear combination (in the sense of Minkowski) of the convex sections $A \cap \Gamma_a$ and $A \cap \Gamma_b$ (in any affine chart $\mathbb{R}P^n \setminus \Gamma_c$, $c \notin [a, d]$), so it is convex.

Let us prove that the complement to the projection of the set $S_{[a,b]}(A)$ from an arbitrary center $L_1 \subset L$, where L_1 is a hyperplane in L, is a convex open set. Consider the projection $\pi(A)$ of the set A on the projective plane $\mathbb{R}P^n/L_1$. The set A is L-convex-concave, so the complement B to the projection $\pi(A)$ is an open convex set containing the marked point $\pi(L)$. The plane $\mathbb{R}P^n/L_1$ contains two lines, $l_a = \pi(\Gamma_a)$ and $l_b = \pi(\Gamma_b)$ passing through the point $\pi(L)$, the half-plane $l^1_{[a,b]} = \pi(\Gamma^1_{[a,b]})$, and the complementary half-plane $l^2_{[a,b]} = \pi(\Gamma^2_{[a,b]})$.

From the definition of the set $S_{[a,b]}(A)$, we see that the complement $B_{[a,b]}$ to its projection $\pi(S_{[a,b]}(A))$ has the following structure.

(1) The set $B_{[a,b]}$ contains the point $\pi(L)$;
(2) Consider two closed triangles with vertices at the point $\pi(L)$ lying in $l^1[a, b]$and such that one side of each of them is the segment lying inside $l^1[a, b]$ and joining the intersection points of lines l_a and l_b with the boundary of the domain B (see Fig. 3). The set $B_{[a,b]} \cap l^1[a, b]$ is the union of these triangles from which the sides described above are removed;
(3) The set $B_{[a,b]} \cap l^2[a, b]$ coincides with the set $B \cap l^2[a, b]$.

From this description, we see that the set $B_{[a,b]}$ is convex and open. □

If two segment $[a, b]$ and $[c, d]$ on the line $\mathbb{R}P^n/L$ do not have common interior points, then the surgeries $S_{[a,b]}$ and $S_{[c,d]}$ commute. We can divide the line $\mathbb{R}P^n/L$ into a finite set of segments $[a_1, a_2], \ldots, [a_{k-1}, a_k], [a_k, a_1]$ and apply the surgeries corresponding to these segments to an L-convex-concave set A. As a result, we obtain an L-convex-concave set D such that the sections of D by hyperplanes $\Gamma_{a_1}, \ldots, \Gamma_{a_n}$ coincide with the sections $A \cap \Gamma_{a_i}$ of the set A by the same hyperplanes. For an intermediate value $a_i < a < a_{i+1}$, the section $D \cap \Gamma_a$ coincides with the section by the same hyperplane of the convex hull of the union of the sections $A \cap \Gamma_{a_i}$ and $A \cap \Gamma_{a_{l+1}}$ in the affine chart $\mathbb{R}P^n \setminus \Gamma_c$ (where c is any point of the line $\mathbb{R}P^n/L$ not belonging to the segment $[a_i, a_{i+1}]$).

The second surgery: $\dim L = 1$**.** A one-dimensional space L corresponds to an $(n-2)$-dimensional bundle of two-dimensional planes containing the line L. Fix two points a and b and a segment $[a,\, b]$ on the line L being one of the two segments into which the points a and b divide the line L. For any L-convex-concave set A and the segment $[a,\, b] \subset L$, we construct a new L-convex-concave set $P_{[a,b]}(A)$. The section of $P_{[a,b]}(A)$ by any two-dimensional plane N, $N \supset L$, depends only on the section of the set A by this plane N.

We define first an operation $F_{[a,b]}$ on two-dimensional convex sets. This operation $F_{[a,b]}$ transforms plane sections $A \cap N$ of the set A into plane sections $P_{[a,b]}(A) \cap N$ of the set $P_{[a,b]}(A)$.

Consider a two-dimensional projective plane N with a distinguished projective line L and a segment $[a,\, b] \subset L$. Let $\Delta \subset N$ be any closed convex subset of the plane N not intersecting the line L.

By definition, the operation $F_{[a,b]}$ transforms a set $\Delta \subset N$ into the smallest convex set $F_{[a,b]}(\Delta)$ containing the set Δ and pointed relative to the segment $[a,\, b]$.

A more explicit description of the set $F_{(a,b)}(\Delta)$ is as follows.

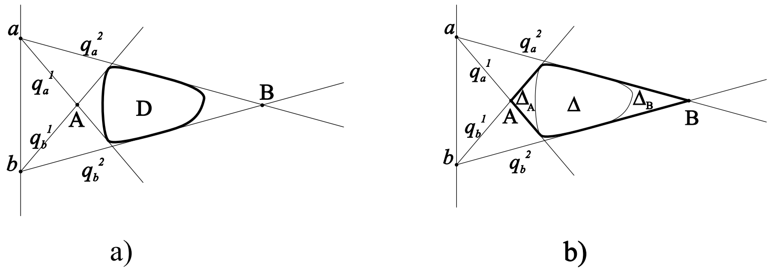

FIGURE 4. (a) The set Δ and its tangents passing through the points a and b; (b) The set $F_{(a,b)}(\Delta)$.

Let us draw four tangents, q_a^1, q_a^2 and q_b^1, q_b^2, to the set Δ passing through the points a and b, respectively (see Fig. 4). The convex quadrangle Δ_1 in the affine plane $N \setminus L$ with sides on the lines q_a^1, q_a^2 and q_b^1, q_b^2 has exactly two vertices A and B such that the support lines to the quadrangle Δ_1 at these vertices do not intersect the segment $[a,\, b]$. The vertex A corresponds to a curvilinear triangle Δ_A with two sides lying on the two sides of the quadrangle Δ_1 adjacent to the vertex A. The third side of Δ_A coincides with the part of the boundary of the set Δ visible from the point A.

A similar curvilinear triangle Δ_b corresponds to the vertex B. Evidently, the set $F_{[a,b]}(\Delta)$ coincides with the set $\Delta_A \cup \Delta \cup \Delta_B$.

Now, we can define the set $P_{[a,b]}(A)$.

Definition. For any L-convex-concave subset A of $\mathbb{R}P^n$, where $\dim L = 1$ and for any segment $[a,\, b]$ of the line L, we define the set $P_{[a,b]}(A)$ by the following condition: the section $P_{[a,b]}(A) \cap N$ of this set by an arbitrary two-dimensional plane N containing L is obtained from the section $A \cap N$ of the set A by applying the operation $F_{[a,b]}$ in the plane N: $P_{[a,b]}(A) \cap N = F_{[a,b]}(A \cap N)$.

Theorem. *For any L-convex-concave set A, where $\dim L = 1$, and any segment $[a,\, b] \subset L$ on the line L, the set $P_{[a,b]}(A)$ is L-convex-concave.*

Proof. Each L-convex-concave set A in $\mathbb{R}P^n$ corresponds to its L-dual $D = (A_L^\perp)$ in the dual projective space $(\mathbb{R}P^n)^*$. The set D is an L^*-convex-concave set, and $\dim L^* = n - 2$. The line L is dual to the set of parameters $(\mathbb{R}P^n)^*/L^*$. The segment $[a,\, b] \subset L$ corresponds to the dual interval $\langle a^*,\, b^* \rangle \subset (\mathbb{R}P^n)^*/L^*$. We use the segment $[a^*,\, b^*] \subset (\mathbb{R}P^n)^*/L^*$ and the L^*-convex-concave set $D = A_L^\perp$ to define a new L^*-convex-concave set $S_{[a^*,b^*]}(D)$. To prove the theorem, it is sufficient to check that the set $P_{[a,b]}(A)$ is L^*-dual to the set $S_{[a^*,b^*]}(D)$, where $D = A_L^\perp$. This is proved below. □

Proposition. *The set $P_{[a,b]}(A)$ is L^*-dual to the set $S_{[a^*,b^*]}(D)$.*

Proof. We have proved that, if the set D is L^*-convex-concave, then $S_{[a^*,b^*]}(D)$ is also L^*-convex-concave and described how to obtain the plane projections of the set $S_{[a^*,b^*]}(D)$ from the plane projections of the set D.

Consider the sets $D_{L^*}^\perp = A$ and $S_{[a^*,b^*]}(D)_{L^*}^\perp$ L^*-dual to D and $S_{[a^*,b^*]}(D)$, respectively. The plane projections of the sets D and $S_{[a^*,b^*]}(D)$ are dual to the plane sections of the sets A and $(S_{[a^*,b^*]}(D))_{L^*}^\perp$. Looking at the plane pictures, we see that the sections of the set $(S_{[a^*,b^*]}(D))_{L^*}^\perp$ are obtained from the sections of the set A by the surgery $F_{[a,b]}$. Therefore, $(S_{[a^*,b^*]}(D))_{L^*}^\perp = P_{[a,b]}(A)$. □

If the complements $\langle a,\, b\rangle_0$ and $\langle c,\, d\rangle_0$ to the segments $[a,\, b]$ and $[c,\, d]$ do not intersect, then the operations $P_{[a,b]}$ and $P_{[c,d]}$ commute. Let us divide the line L into a finite number of intervals $\langle a_1,\, a_2\rangle_0, \ldots, \langle a_{k+1},\, a_1\rangle_0$ complementary to segments $[a_1,\, a_2], \ldots, [a_{k+1},\, a_1]$ (the segments intersect each other) and apply the operations $P_{[a_i,a_{i+1}]}(A)$ corresponding to all these segments to the L-convex-concave set A. As a result, we obtain an L-convex-concave set D whose section by any two-dimensional plane N containing the line L is a polygon with $2k$ sides circumscribed about the section $A \cap N$ (some of the sides of the resulting polygons may degenerate into points). To each point a_i, there correspond two parallel sides of the polygon which pass through the point a_i and lying on the support lines to the section $(A \cap N)$.

Remark. To a three-dimensional set $A \subset \mathbb{R}P^3$ L-convex-concave with respect to a line L, both surgeries can be applied, since $\dim L = 1 = n - 2$ for $n = 3$. Let $[a,\, b]$ be a segment on the line L, and let $[c,\, d]$ be a segment on the line $\mathbb{R}P^3/L$. Then, as can easily be proved, the surgeries $P_{[a,b]}$ and $S_{[c,d]}$ commute.

A space intersecting support half-planes to sections. As above, let A be an L-convex-concave subset of $\mathbb{R}P^n$, where $\dim L = 1$. Consider the following problem. Suppose given a certain set $\{N_\alpha\}$, $\alpha \in I$, of two-dimensional planes containing the line L, and suppose that, on each affine plane $N_\alpha \setminus L$, some half-plane $p_\alpha^+ \subset N_\alpha$ supporting to the convex section $N_\alpha \cap A$ is fixed. We want to find an $(n-2)$-dimensional subspace of $\mathbb{R}P^n$ intersecting all the half-planes p_α^+, $\alpha \in I$.

Theorem. *Suppose that the set $Q = \{\partial p_\alpha^+ \cap L\}$, where $\alpha \in I$and ∂p_α^+ denotes the boundary line of the half-plane p_α^+ supporting for the section $N_\alpha \cap A$ of an L-convex-concave set $A \subset \mathbb{R}P^n$, contains at most four points. Then there exists an*

$(n-2)$-dimensional subspace of $\mathbb{R}P^n$ intersecting all the supporting half-planes p_α^+, $\alpha \in I$.

Proof. Suppose that the set Q contains exactly four points $a_1, \ldots, a_4$ (otherwise, we add a necessary amount of some other points to Q). The points a_i divide the projective line into four segments $\langle a_1, a_2\rangle$, $\langle a_2, a_3\rangle$, $\langle a_3, a_4\rangle$, $\langle a_4, a_1\rangle$. Let $I_1, \ldots, I_4$ denote the complementary segments (these segments intersect each other). We apply the four surgeries P_{I_i} to the set A and denote the resulting set by D.

By the very definition of the set D, the half-planes $p_\alpha^+ \subset N_\alpha$ are supporting half-planes for the sections $D \cap N_\alpha$, so any space lying inside D intersects the half-planes p_α^+. According to the theorem of Section 5 there exists an $(n-2)$-dimensional subspace of $\mathbb{R}P^n$ lying inside the set D. This space intersects all the half-planes π_α^+. □

References

[Ar1] V. I. Arnold, *Selecta* — 60, FAZIS, Moscow, 1997 (Russian). MR **99j:**01025

[Ar2] V. I. Arnold, *Arnold's problem*, FAZIS, Moscow, 2000 (Russian). MR **2002e:**58001

[Br] F. E. Browder, *A new generation of the Schauder fixed point theorem*, Math. Ann. **174** (1967), 285–290. MR 36 #6991

[He1] E. Helly, *Über Systeme Abgeschossener Mengen mit gemeinschlaftlichen Punkten*, Monatsh. Math. **37** (1930), 281–302.

[He2] E. Helly, *Über Mengen konvexen Korper mit gemeinschaftlichen Punkten*, Jber. Deutsch. Math.-Verein. **32** (1933), 175–186.

[KhN1] A. Khovanskii and D. Novikov, *Convex-concave body in $\mathbb{R}P^3$ contains a line*, GAFA **13** (2003), 1082–1118.

[KhN2] A. Khovanskii and D. Novikov, *On affine hypersurfaces with everywhere nondegenerate second quadratic form*, to appear in J. Differential Geom.

[PKh] A. V. Pukhlikov and A. G. Khovanskii, *Finitely additive measures of virtual polyhedra*, Algebra i Analiz **4** (1992), no. 2, 161–185 (Russian), English translation in: St. Petersburg Math. J. **4** (1993), no. 2, 337–356. MR **93i:**52024

[Vi] O. Y. Viro, *Some integral calculus based on Euler characteristic*, Topology and geometry — Rohlin Seminar, Lecture Notes in Math., vol. 1346, Springer, Berlin, 1988, pp. 127–138. MR **90a:**57029

Department of Mathematics, Toronto University, Toronto, Canada
E-mail address: askold@math.toronto.edu

Department of Mathematics, Purdue University, West Lafayette IN
E-mail address: dmitry@math.purdue.edu