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L-CONVEX-CONCAVE BODY IN $\mathbb{R}P^3$ CONTAINS A LINE

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Abstract

We define a class of L-convex-concave subsets of $\mathbb{R}P^3$, where L is a projective line in $\mathbb{R}P^3$. These are sets whose sections by any plane containing L are convex and concavely depend on this plane. We prove a version of Arnold's conjecture for these sets, namely we prove that each such set contains a line.

1 Introduction

A classical definition of the convex domain in an affine space admits a natural projective analogue. Namely, a subset of $\mathbb{R}P^n$ is called convex if it does not intersect some hyperplane and is convex in the complement to it in the usual sense if one identifies this complement with an affine space.

The main object of interest in this paper is a class generalizing the class of convex subsets of $\mathbb{R}P^n$, the class of the so-called L-convex-concave subsets of $\mathbb{R}P^n$. The letter L denotes here a projective subspace of $\mathbb{R}P^n$, and if L is a hyperplane then we get the usual convex sets. Motivation for the definition came initially from attempts to prove or disprove the Arnold conjecture (see below). However, it turned out that this class is quite interesting in itself. Many properties and operations possible for convex sets can be generalized for the class of L-convex-concave sets, including duality, affine structure, and others, see [KN1]. What is remarkable, however, is the fact that, unlike the convex sets, there is a difference between affine and projective definitions of L-convex-concave sets. In other words, a similar affine generalization of the notion of the convex set gives an essentially wider class with less remarkable properties, and Theorem 1 of this paper together with an example from [KN2] can be considered as a manifestation of this difference.

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1.1 Definitions of L-convex-concave subsets of $\mathbb{R}P^n$. In the very definition of convex subsets of a projective space a hyperplane appears. Similarly, the class of L-convex-concave subsets of $\mathbb{R}P^n$ depends on the choice of L, a projective subspace of $\mathbb{R}P^n$. Here is the definition.

DEFINITION 1. Let L be a fixed proper projective subspace of $\mathbb{R}P^n$. A closed set $A \subset \mathbb{R}P^n$ is L-convex-concave if

1. $A \cap L = \emptyset$,
2. for any projective subspace $N \subset \mathbb{R}P^n$ of dimension $\dim L + 1$ and containing L the intersection $A \cap N$ is convex,
3. for any projective subspace $T \subset L$ of dimension $\dim L - 1$ the complement to the image of $\pi(A)$ under projection $\pi : \mathbb{R}P^n \setminus T \to \mathbb{R}P^n/T \cong \mathbb{R}P^{n-\dim L}$ is an open convex set (if $\dim L = 0$ then we take an identity map as the projection π).

For subspaces L of different dimensions we get different classes. Here are some examples of L-convex-concave sets. For $\dim L = n - 1$ the first two requirements are exactly the definition of a convex subset of $\mathbb{R}P^n$, and the third means that it is nonempty, so nonempty convex sets are indeed a particular case of L-convex-concave subsets of $\mathbb{R}P^n$.

For $\dim L = 0$ the second condition means that the intersection of A with any projective line containing the point L is a segment, and that the complement to A is an open convex set (in this case π is just an identity mapping). In other words, A is a complement to an open convex set containing L.

Domains bounded by a quadric are another example of L-convex-concave sets. Let $Q(x,x)$ be a quadratic form positively defined on L and negatively defined on some L', a projective subspace of dimension $n - \dim L - 1$ not intersecting L. Then the set $A = \{Q \leqslant 0\}$ is L-convex-concave: the sections and projections are both bounded by spheres.

For any L a projective subspace A of maximal possible dimension not intersecting L (namely of dimension equal to $n - \dim L - 1$) is a L-convex-concave set. Indeed, the intersections $A \cap N$ are just points (so convex), and the projection $\pi(A)$ will be a hyperplane in $\mathbb{R}P^n/T \cong \mathbb{R}P^{n-\dim L}$, so its complement is an affine space $R^{n-\dim L}$.

This example can be considered as a limit case of the previous example: it corresponds to $Q \equiv 0$ on $A = L'$.

1.2 Main theorem. The main conjecture about the L-convex-concave sets claims that the previous example is the backbone of any L-convex-concave set.

CONJECTURE 1 (Main Conjecture). *Any L-convex-concave domain $A \subset \mathbb{R}P^n$ contains a projective subspace of dimension equal to $n - \dim L - 1$.*

Evidently, this conjecture is true for a particular case of nonempty convex sets: in this case $n - \dim L - 1 = 0$, and conjecture claims that the set is nonempty. The first nontrivial case is of $n = 3$ and $\dim L = 1$. The main result of this paper claims that in this case the Main Conjecture is true.

Theorem 1 (Main Theorem). *Any L-convex-concave set $A \subset \mathbb{R}P^3$, $\dim L = 1$, contains a projective line.*

1.3 Arnold's conjecture. The main conjecture about L-convex-concave sets is closely connected to the Arnold conjecture about quasi-convex hypersurfaces in $\mathbb{R}P^n$, see [A]. The conjecture is motivated by a theorem about locally convex hypersurfaces proved in the same paper. Here is the statement of this theorem. Let M be a connected closed hypersurface M without a boundary embedded to $\mathbb{R}P^n$. Suppose that the second fundamental form of M is everywhere negatively defined (it means that in some affine coordinates M is locally defined as $x_n = -x_1^2 - \ldots - x_{n-1}^2 +$ higher order terms). The theorem proved in [A] claims that M bounds a convex body, i.e. that there is a hyperplane H not intersecting M and M bounds a convex body in the affine space $\mathbb{R}P^n \setminus H$.

Arnold (in [A]) conjectured that an analogue of this fact holds for any hypersurface with an everywhere non-degenerate second fundamental form. We will say that a quadratic form in $\mathbb{R}^{n-1}$ has signature $(n-k-1, k)$ if its restriction to some k-dimensional linear subspace is negatively defined and its restriction to some $n-k-1$-dimensional linear subspace is positively defined.

CONJECTURE 2 (Arnold's conjecture). *Consider a domain $U \subset \mathbb{R}P^n$ bounded by a connected smooth hypersurface B without boundary. Suppose that the second fundamental form of B with respect to the outward normal is non-degenerate at any point of B and has signature $(n-k-1, k)$ (necessarily the same for all points). Then there exist a projective subspace L^k of dimension k contained in U and a projective subspace L^{n-k-1} of dimension $n-k-1$ not intersecting U.*

For $k = n-1$ this conjecture becomes the aforementioned theorem in [A] about locally convex hypersurfaces.

Another example is the domains bounded by quadrics considered above as an example of an L-convex-concave sets. Their boundary have the signature of the second quadratic form equal to $(n - \dim L - 1, \dim L)$. The conjecture is true for these domains: one can take L and L' as the required projective subspaces.

The fact that the domains bounded by quadrics are both L-convex-concave and have constant signature of the second quadratic form of the boundary is not a coincidence. In fact, L-convex-concave domain with smooth boundary will necessarily have constant signature of the second quadratic form of the boundary (namely equal to $(n - \deg L - 1, \deg L)$) as soon as it is non-degenerate. Moreover, any L-convex-concave set after a small perturbation becomes a L-convex-concave domain with smooth boundary and non-degenerate second quadratic form, see [KN2], so the main conjecture is a consequence of the Arnold conjecture.

Arnold's conjecture is stronger than our main conjecture since it concerns a wider class of domains: not all domains with constant signature of the second quadratic form are L-convex-concave for some L. The difference between the two conjectures is twofold. First, in the very definition of the L-convex-concave domain we postulate the existence of one of the subspaces whose existence is claimed in the Arnold conjecture. Second, in the definition of L-convex-concave domains we suppose that all their sections by subspaces containing L as a hyperplane are convex.

1.4 Affine L-convex-concave sets and Arnold's conjecture. One can try to look on the Arnold conjecture from an affine point of view: take any affine chart $\mathbb{R}^n$ of $\mathbb{R}P^n$, and reformulate the conjecture in affine terms. The Arnold conjecture then claims that domain in $\mathbb{R}P^n$ bounded by a closed connected hypersurface B without boundary with constant signature of its second quadratic form necessarily contains an affine subspace, and the projective origins of the hypersurface will be translated to some conditions on the asymptotical behavior of B.

In [KN2] we approach the Arnold conjecture from this affine point of view. The main goal was to clarify the role of the projective space in the conjecture.

First we consider domains asymptotically approaching the quadratic cone $K = \{x_1^2 + ... + x_k^2 - x_{k+1}^2 - ... - x_n^2 = 0\}$ as $|x| \to +\infty$. We prove Arnold's conjecture in this case.

Next we considered a slightly different asymptotic behavior. Namely, we considered domains bounded by a surface asymptotically approaching a union $K' \subset \mathbb{R}^3$ of moved apart halves of the standard cone $K = \{x^2 + y^2 = z^2\} \subset \mathbb{R}^3$(e.g. $K' = \{(x, y, z) \mid x^2 + y^2 = (|z| - 1)^2, |z| \geqslant 1\}$). In this case the Arnold conjecture is false: we constructed an example of such a domain not containing any lines.

These two facts strengthen the importance of the projective settings in Arnold's conjecture. Indeed, these two asymptotic conditions, quite similar from an affine point of view, imply a different form of projective closure of the domain in $\mathbb{R}P^3$: in the first case the closure has locally smooth boundary and in the second case the closure will be non-smooth and, more importantly, locally convex at the points of intersection with the infinite plane.

The example constructed in [KN2] gave birth to the definition of L-convex-concave sets, since it was constructed using *affine convex-concave* sets. Here is the definition: a domain $A \subset \mathbb{R}^3$ is called affine convex-concave if all its sections by horizontal planes are convex, compact and concavely depend on the plane. The last condition simply means that for any three horizontal sections the middle one lies inside the convex hull of the union of two external sections. As in the projective case, as soon as the second quadratic form of a boundary of an affine convex-concave set is defined (i.e. when this boundary is smooth) and nondegenerate, it has signature $(1,1)$, so satisfies the conditions of the Arnold conjecture. A counterexample was therefore constructed by a smoothening in the class of the affine convex-concave sets of a strip – a very degenerate affine convex-concave set not containing lines.

Any L-convex-concave subset of $\mathbb{R}P^3$, $\dim L = 1$, is affine convex-concave in any chart for which L is an infinite horizontal line. However, the opposite is not true: by virtue of Theorem 1 the closure in $\mathbb{R}P^3$ of the example above is not L-convex-concave subset.

1.5 Structure of the paper. The proof of Theorem 1 belongs in fact to the realm of convex geometry. It heavily exploits the two fundamental theorems of convex geometry: Helly's theorem and the Browder theorem. The proof is partly guided by the general ideology of the Chebyshev best approximation. In particular, one of the key ingredients of the proof is an analogue of the Chebyshev alternance, see Lemma 7 and Theorem 10.

Further we will consider only bodies L-convex-concave with respect to some fixed once and forever projective line L. Also, we will use an equivalent definition of a convex-concave set. Namely, in [KN1] it is shown that the L-convex-concave subsets of $\mathbb{RP}^3$ can be characterized in the following way.

DEFINITION 2. A body $\mathbb{B} \in \mathbb{RP}^3$ is called *L-convex-concave* with respect to a line L (further called *infinite horizontal line*) not intersecting $\mathbb{B}$ if

- sections of $\mathbb{B}$ by planes passing through this line (further called *horizontal planes*) are all compact and convex and

- for any three such horizontal sections, through any point of any of them passes a line intersecting two another.

For convenience we will suppress the letter "L" and will use the term *convex-concave* for the L-convex-concave bodies.

The rest of the paper contains only the proof of the Theorem 1 and is organized as follows. In §2 we show that it is enough to prove that for any five sections of a convex-concave set by planes containing L there is a line intersecting all of them, see Theorem 2. This is a standard application of the Helly theorem. From the other hand, using Browder theorem, we prove that for any four sections one can find a line intersecting all of them, see Theorem 4.

Starting from §3 we are dealing with five fixed sections of a convex-concave body. The general idea is simple. Fix an Euclidean metric on some affine cart in $\mathbb{R}P^3$ containing all five sections and not containing L and take a line closest to these five sections (the Chebyshev line). Our goal is to prove that one can always find a line which lies closer to these five sections, unless the Chebyshev line intersects all five sections.

More exact, in §3 we introduce the Euclidean metric, define the Chebyshev line and establish its basic properties. On planes containing sections arise five half-planes with the property that any line lying closer to five sections than the Chebyshev line should intersect all these half-planes. The opposite is almost true. Namely, any line intersecting these half-planes (further called *good deformation*) produces a line closer to the sections than the Chebyshev line, see Lemma 5. So all we need to prove is the existence of a line intersecting these five half-planes, which depends on the projective properties of their mutual position only. These properties are the main object of further investigations.

At this stage a split occurs. We impose a condition of genericity on the collection of these half-planes (namely, their boundaries should be pairwise non-parallel) and deal further with non-degenerate cases only. In degenerate cases existence of the good deformation follows from Theorem 4 due to a remarkable self-duality of the condition of L-convex-concavity, see §3.4 and [KN1].

In §4 and §5 we investigate combinatorial properties of a collection of five half-planes corresponding to a Chebyshev line, forgetting for a moment the convex-concavity condition. In other words, we consider a more general problem of properties of a line closest to five convex figures on five parallel planes. This reduces to a purely combinatorial problem about possible arrangements of rooks on a chess board. We find an equivalent of the classical condition of Chebyshev alternance for our situation. Namely, only six possible combinatorial types of collections of half-planes are possible, see Theorem 10.

In §6 for each of these six types we prove existence of a good deformation using the convex-concavity condition. More exact, each of these combinatorial types have some continuous parameters (namely distances between sections and double ratios of angles between their boundaries). If a configuration of half-planes arose from a Chebyshev line as above, then these parameters should satisfy some inequalities. In other words, only part of the space of parameters corresponds to Chebyshev alternances. It turns out that configurations of half-planes arising from sections of a convex-concave body belonging to the complement to this part.

More exactly, using the combinatorial properties of each case and Theorem 11, we are able to prove existence of a line intersecting four of the half-planes in a some particular sectors. These sectors are chosen in such a way that the line intersecting them should necessarily intersect the fifth half-plane and the existence of a good deformation follows.

2 Applications of Helly's Theorem and Browder's Theorem

In this section we first introduce a linear structure on the set of all lines not intersecting the line L. We prove that the Theorem 1 follows from the fact that for any five sections of a convex-concave body there is a line intersecting all of them. Another result claims that for any *four* sections there is a line intersecting all of them.

2.1 Affine structure on the set of all non-horizontal lines. We call a plane *horizontal* if it contains the line $L \subset \mathbb{R}P^3$ (further called the infinite horizontal line). A line is called *non-horizontal* if it doesn't intersect L.

We choose coordinates (x, y, z) in a complement to some horizontal plane in such a way that the infinite horizontal line lies in the projective plane $\{z = 0\}$. In these coordinates the non-horizontal lines have a parameterization of the type $x = az + b$, $y = cz + d$. This correspondence $\{\text{non-horizontal line}\} \to (a, b, c, d)$ defines coordinates on the set $\mathbb{U} \cong \mathbb{R}^4$ of all non-horizontal lines.

An affine structure introduced on $\mathbb{U}$ by these coordinates is compatible with the affine structure in horizontal planes: intersection of a convex combination of two lines with a horizontal plane is a convex combination (with the same coefficients) of intersections of these two lines with this plane. Therefore the affine structure defined by these coordinates is independent of the choice of coordinates and depends on the choice of the infinite horizontal line only (however, the linear structure, i.e. the line with coordinates $(0,0,0,0)$ (=z-axis), can be chosen arbitrarily).

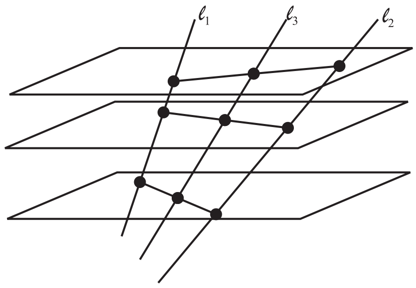

Figure 1: ℓ_3 is a linear combination of ℓ_1 and ℓ_2

Denote by U_t the set of all non-horizontal lines intersecting a horizontal section $S_t = \mathbb{B} \cap \{z = t\}$ of a projective convex-concave body $\mathbb{B}$. From the last remark we immediately see that

LEMMA 1. *U_t is closed and is convex in the coordinates introduced above.*

The inverse is also true. Namely, for any horizontal plane $\{z = t\}$ there is a map $\phi_t : \mathbb{U} \to \{z = t\}$ mapping a non-horizontal line to its point of intersection with this plane.

LEMMA 2. *This map preserves convexity, i.e the image of a convex subset of $\mathbb{U}$ is again a convex set.*

2.2 Non-horizontal lines and sections of a convex-concave body.

2.2.1 Five sections: Helly's theorem. Let $\mathbb{B}$ be a convex-concave subset of $\mathbb{R}P^3$.

Theorem 2. *Theorem 1 follows from the following claim:*

$$\forall\, t_1, t_2, t_3, t_4, t_5 \in \mathbb{R} \qquad \bigcap_{i=1}^{5} U_{t_i} \neq \emptyset\,.$$

In other words, it is enough to prove that for any five horizontal sections S_i of $\mathbb{B}$ there exists a line intersecting all of them.

Proof. Indeed, Theorem 1 is equivalent to $\bigcap_t U_t \neq \emptyset$. Since U_t are convex subsets of $U \cong \mathbb{R}^4$, the claim is almost a particular case ($n = 4$) of the classical Helly theorem:

Theorem 3 (Helly's theorem, see [H1,2]) **.** *Intersection of a finite family of closed convex sets in $\mathbb{R}^n$ is nonempty if and only if intersection of any $n+1$ of them is nonempty.*

The only problem is that the family U_t is not finite. However, one can circumvent this technicality in a standard way using the fact that

LEMMA 3. *Intersection of $U_{t_1} \cap U_{t_2}$ is compact for any $t_1 \neq t_2$.*

Indeed, any line belonging to $U_{t_1} \cap U_{t_2}$ is uniquely defined by its points of intersection with these two sections, so $U_{t_1} \cap U_{t_2}$ is homeomorphic to $S_{t_1} \times S_{t_2}$, which is compact.

So, take a compact $K = U_1 \cap U_0$ and consider a family of sets $\widetilde{U}_t = K \setminus U_t$. These sets are relatively open in K. We want to prove that $\cup_t \widetilde{U}_t \neq K$. If not, then $\widetilde{U}_t$ is a covering of K, so we can take a finite family $\{\widetilde{U}_{t_i}\}$ covering K. This means that the intersection of a *finite* family consisting of the corresponding U_{t_i} and U_1 and U_0 will be empty. This is impossible by Helly's theorem if intersection of any five of U_t is nonempty. □

2.3 Four sections: Browder's theorem.

It turns out that the convex-concavity condition (even the affine one, see §1.4 for a definition) guarantees existence of a line passing through any four sections. We will prove this with slightly more general assumptions.

Theorem 4. *Let A, B, C, D be four compact convex non-empty sets in $\mathbb{R}^n$ satisfying the following conditions:*

1. *$A \subset \{x_n = t_1\}$, $B \subset \{x_n = t_2\}$, $C \subset \{x_n = t_3\}$, $D \subset \{x_n = t_4\}$, where t_i are pairwise different;*
2. *through any point of B a line passes intersecting both A and C, and*
3. *through any point of C a line passes intersecting both B and D.*

Then there exists a line intersecting all four bodies.

REMARK 1. Here we use only some of the conditions provided by convex-concavity.

We will use a Browder theorem – a fixed-point theorem for upper semi-continuous set-valued mappings, see [B]. Here is the statement of this theorem.

Let $f : X \to Set(X)$ be a mapping from X to the set of all subsets of X.

DEFINITION 3. f is called *upper semi-continuous* on X if for any $x_0 \in X$ and any open set G containing $f(x_0)$ there exists a neighborhood U of x_0 such that $f(x) \subset G$ for all $x \in U$.

REMARK 2. For single-valued maps this property means continuity.

Theorem 5 (Browder's theorem, see [B]). *Let X be a non-empty compact convex set in a real, locally convex, Hausdorff topological vector space E. Let f be an upper semi-continuous set-valued mapping defined on X such that for each $x \in X$, $f(x)$ is a non-empty closed convex subset in X. Then there exists a point $\hat{x} \in X$ with $\hat{x} \in f(\hat{x})$.*

We will apply Theorem 5 to the composition $f : B \to CSet(B)$ of the tautological map $B \to CSet(B)$ and two maps $h_1 : CSet(B) \to CSet(C)$ and $h_2 : CSet(C) \to CSet(B)$, where $CSet(B)$ and $CSet(C)$ are sets of all compact convex subsets of B and C correspondingly. Namely, for $U \subset B$ we define $h_1(U) \subset C$ as a set of all points of C which lie on a line intersecting both A and U. Similarly, for $V \subset C$ we define $h_2(V) \subset B$ as set of all points of B which lie on a line intersecting both D and V. These maps are completely defined by their restrictions to the one-point subsets of B and C correspondingly, namely $h_i(U) = \cup_{x\in U} h_i(\{x\})$.

Check first that our result indeed follows from Theorem 5. Suppose that for some $x \in B$ we have $x \in f(x)$. This means that $x \in h_2(y)$ for some point $y \in h_1(\{x\})$. By definition of h_i, this means that the line passing through x and y intersects both A and D. □

We have to check that $f(x)$ satisfies conditions of Theorem 4.

By convex-concavity $f(x)$ is non-empty for all $x \in B$.

LEMMA 4. *$f(x) = h_2(h_1(\{x\}))$ is upper semi-continuous.*

We will prove that both h_1 and h_2 are upper semi-continuous in the sense defined below, and the claim will follow from the fact that the composition of upper semi-continuous maps is again upper semi-continuous. Denote by $N_\delta(U) = \{x | \operatorname{dist}(x, U) < \delta\}$ the δ-neighborhood of U.

LEMMA 5. *Mapping h_1 is upper semi-continuous in the following sense: for any $U \in CSet(B)$ and any $\epsilon > 0$ there exist a $\delta > 0$ such that if $U' \subset N_\delta(U)$ then $h_1(U') \subset G = N_{2\epsilon}(h_1(U))$. The mapping h_2 is also upper semi-continuous.*

Proof. We prove it for h_1, and the proof for h_2 differs by notation only. By definition $h_1(U) = \cup_{x\in U} h_1(\{x\})$. Therefore by compactness of U it is enough to prove that for any $b \in B$ and any $\epsilon > 0$ there is a $\delta > 0$ such that if $\operatorname{dist}(b', b) < \delta$ then $h_1(b') \subset N_\epsilon(h_1(b))$.

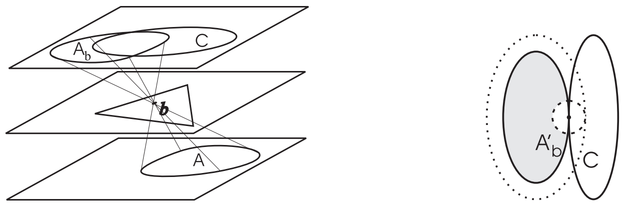

Figure 2: Construction of $h_1(b)$ and upper semi-continuity of $h_1(U)$

Note that $h_1(b) = C \cap A_b$, where A_b is a compact upper semi-continuously depending on b in Hausdorff metric (in fact A_b and $A_{b'}$ differ by a parallel translation only).

The claim follows from the fact that an intersection of a compact with another compact upper semi-continuously depending on parameters depends upper semi-continuously on parameters. Let's prove this fact. Let $V = h_1(b)$, and $A'_b = A_b \setminus N_\epsilon(V)$. Let $0 < \alpha < \min(\epsilon, \operatorname{dist}(A'_b, C))$. For b' close enough to b we have $A_{b'} \subset N_\alpha(A_b)$ and

$$\begin{aligned} h_1(b') = C \cap A_{b'} \subseteq C \cap N_\alpha(A_b) &\subseteq \big(C \cap N_\alpha(A'_b)\big) \cup \big(C \cap N_\alpha(N_\epsilon(V))\big) \\ &\subseteq C \cap N_{\alpha+\epsilon}(V) \end{aligned}$$

and the last one is a subset of G.

The second inclusion is true by upper semi-continuous dependence of A_b on b, the third is true since $A_b \subset A'_b \cup N_\epsilon(V)$, the fourth is true since $C \cap N_\alpha(A'_b) = \emptyset$ by choice of α and the last one is true since $\alpha + \epsilon < 2\epsilon$. □

To satisfy the last condition of Theorem 5 we have to check that $f(x)$ is a closed convex subset of B.

LEMMA 6. *$h_i(U)$ is compact convex set as soon as U is compact convex set.*

Proof. Indeed, the set of lines intersecting both U and A is convex (as intersection of two convex closed sets) and compact (since a line is defined by its two points of intersection with U and A, which are both compact), so the set of points of intersections of these lines with $\{x_n = t_3\}$ is also convex and compact. But $h_1(U)$ is exactly the intersection of this set with C, so it is also convex and compact. □

REMARK 3. From a Leray theorem and the previous result we get that the set of non-horizontal lines intersecting at least one of the chosen five sections is homotopically equivalent to a ball or to a sphere according to the existence or nonexistence of a line passing through all five sections. We know that there exist affine convex-concave bodies (see the introduction and [KN2]) without a line inside, so the case of a sphere is possible. This sphere divides the set of all non-horizontal lines into two connected parts. As a corollary we see that for some five sections of these affine convex-concave body (in our example in [KN2] these are just line segments) there is a line not intersecting them which cannot be moved to infinity without intersecting the sections.

3 Chebyshev Line

By the previous section all we need to prove is that through any five horizontal sections of the convex-concave body passes a line. We fix them from now on. We choose a sixth horizontal plane L (not containing sections), choose affine coordinates in $\mathbb{R}^3 \cong \mathbb{R}P^3 \setminus L$ and, using a standard scalar product, introduce a metric on horizontal planes. Using this metric we define a Chebyshev line – a line minimizing the maximal distance from its point of intersection with a plane of the section to the section. On each plane containing a section we choose a half-plane containing the section with boundary passing through the point of intersection of a Chebyshev line with the plane and perpendicular to the shortest segment joining this point and the section.

In this and the next section we investigate combinatorial conditions imposed on the configuration of these half-planes by the fact that the Chebyshev lines minimizes the maximal distance to the sections.

3.1 A Chebyshev line. Denote by S_1, S_2, S_3, S_4 and S_5 the five sections of a convex-concave body $B \in \mathbb{RP}^3$ cut by five horizontal planes L_i, i.e. $S_i = B \cap L_i$. Choose coordinates $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w})$ in $\mathbb{RP}^3$ in such a way that the infinite horizontal line has equation $\tilde{z} = \tilde{w} = 0$ and $S_i \subset \{\tilde{w} \neq 0\} \cong \mathbb{R}^3$. We take standard coordinates $(x = \tilde{x}/\tilde{w},\ y = \tilde{y}/\tilde{w},\ z = \tilde{z}/\tilde{w})$ in $\{\tilde{w} \neq 0\} \simeq \mathbb{R}^3$. In these coordinates the planes L_i are given by equations $L_i = \{z = t_i\}$. We take metric on L_i induced by a scalar product

$$\big((x_1, y_1, z_1), (x_2, y_2, z_2)\big) = x_1x_2 + y_1y_2 + z_1z_2 \,.$$

Suppose that there is no line intersecting all five sections S_i (otherwise there is nothing to prove).

Definition 4. The (non-horizontal) line ℓ minimizing the

$$\max_{i=1,\dots,5} \operatorname{dist}(\ell \cap L_i, S_i)$$

(where L_i are the horizontal planes containing S_i) will be called a Chebyshev line.

The existence of this line follows from compactness of the sections. Further we will denote $a_i = \ell \cap L_i$ and by $s_i \in S_i$ the point of S_i closest to a_i.

Lemma 7 (Chebyshev property). *The* $\operatorname{dist}(a_i, S_i) = \operatorname{dist}(a_i, s_i)$ *are all equal.*

Proof. Indeed, let one of them, say $\operatorname{dist}(a_1, S_1)$, is smaller than all others and strictly smaller than one of the distances. By Theorem 4 there exists a line ℓ_1 which intersects all four remaining sections. Therefore for small values of ϵ the points of intersections of the line $\ell_\epsilon = (1-\epsilon)\ell + \epsilon\ell_1$ lies closer to S_i than a_i for $i = 2, 3, 4, 5$. From the other hand, $\operatorname{dist}(\ell_\epsilon \cap L_1, S_1)$ changes continuously with ϵ. So for small $\epsilon > 0$ we get $\max_{i=1,\dots,5} \operatorname{dist}(\ell_\epsilon \cap L_i, S_i) < \max_{i=1,\dots,5} \operatorname{dist}(a_i, S_i)$, which contradicts to the Chebyshev property of ℓ. □

Corollary 1. *The Chebyshev line* ℓ *doesn't intersect* S_i *for any* i *if there is no line intersecting all* S_i.

Further, in order to simplify the notation, we will suppose that the coordinates are chosen in such a way that the Chebyshev line coincides with the z axis. Indeed, a linear transformation of the type $(x, y, z) \to (x - (az + b), y - (cz + d), z)$ doesn't change metric in horizontal planes, so the Chebyshev line for the shifted sections will be the shifted Chebyshev line. From the other side, using a transformation of this type we can move any non-horizontal line to the z-axis.

3.2 Five half-planes. The definition of the Chebyshev line ℓ means that one cannot find another line intersecting the planes L_i at points a_i' such that $\operatorname{dist}(a_i', S_i) < \operatorname{dist}(a_i, S_i)$, where $a_i = \ell \cap L_i$. Here we reformulate this condition in different terms.

For each $a_i = \ell \cap L_i$ we can indicate an angle of desirable directions in L_i: if a_i moves in this direction then the $\operatorname{dist}(a_i, S_i)$ decreases. These are directions forming an acute angle with the direction $\overrightarrow{a_i s_i}$. So the half-plane $H_i = \{x \in L_i | (\overrightarrow{a_i x}, \overrightarrow{a_i s_i}) \geqslant 0\}$ arises. The vector $\overrightarrow{a_i s_i}$ is orthogonal to its boundary and is directed inward.

Another description of H_i is as follows: the function $f(x) = \operatorname{dist}(x, S_i)$ is a smooth function everywhere on $L_i \setminus S_i$, so in particular for $x = a_i$. After identification of $T_{a_i} L_i$ and L_i the half-plane H_i is described as $\{df_{a_i}(\cdot) \geqslant 0\}$.

We will need further the following evident statement, see Figure 3:

Lemma 8. *Let* H *be a half-plane in* L_1 *bounded by a line passing through* a_1 *and normal to* $\overrightarrow{a_1 n}$. *Suppose that* $S_1 \subset H$. *Then* $n \in H_1$.

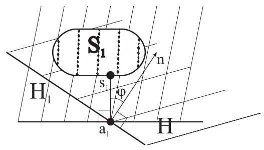

Figure 3: The half-plane H contains s_1, so $n \in H_1$

3.3 Good deformations. Here we describe lines (further called good deformations) whose existence contradicts to the fact that the Chebyshev line ℓ doesn't intersect the sections S_i . Our goal from now is to prove their existence.

LEMMA 9. *If ℓ is the Chebyshev line for S_i and H_i are as above, then there exist no line intersecting interiors of all H_i.*

Proof. Suppose there exists a line $\tilde{\ell}$ intersecting interiors of all H_i. Then z-axis cannot be a Chebyshev line since

$$\max_i \operatorname{dist}\left((\epsilon\tilde{\ell} + (1-\epsilon)\ell) \cap L_i, S_i\right) < \max_i \operatorname{dist}(a_i, S_i)$$

for $\epsilon > 0$ small enough. In other words, moving the Chebyshev line in the direction of $\tilde{\ell}$ in the space of all non-horizontal lines decreases its distance to S_i.

Indeed, all we have to check is that $\frac{d}{d\epsilon}(\operatorname{dist}((\epsilon\tilde{\ell} + (1-\epsilon)\ell) \cap L_i, S_i))|_{\epsilon=0} < 0$, which follows directly from definitions (of H_i and the linear structure on the set of non-horizontal lines). □

In fact one can prove a stronger claim.

DEFINITION 5 (Good deformation). A line ℓ_1 intersecting all H_i and the interior of at least one of them will be called a *good deformation.*

LEMMA 10. *If ℓ is the Chebyshev line for S_i and H_i are as above, then there is no good deformation.*

Proof. The proof uses the same idea as Lemma 7. Suppose that ℓ_1 intersects the interior of H_1 and denote by ℓ_{2345} the line intersecting S_2, S_3, S_4, S_5 (it exists by Theorem 4). Consider the two-parametric family of lines $\lambda\ell_1 + \mu\ell_{2345} + (1-\lambda-\mu)\ell$. The idea is that, in linear approximations, moving ℓ toward ℓ_1 decreases distance to S_1 (while not increasing other distances), and moving ℓ toward ℓ_{2345} decreases distances to all other sections. So some combination of these two movements decreases the maximal distance from the Chebyshev line to sections, which is impossible.

In other words, denote points of intersection of $\lambda\ell_1 + \mu\ell_{2345} + (1-\lambda-\mu)\ell$ with L_i by $b_i^{\lambda,\mu}$. Then $\frac{\partial}{\partial\lambda}|_{\lambda=\mu=0} \operatorname{dist}(b_i^{\lambda,\mu}, S_i)$ are non-positive for $i = 2,3,4,5$, and is strictly negative for $i = 1$. Also $\frac{\partial}{\partial\mu}|_{\lambda=\mu=0} \operatorname{dist}(b_i^{\lambda,\mu}, S_i)$ are strictly negative for $i = 2,3,4,5$. Therefore for some positive c_1, c_2 we have $\frac{\partial}{\partial\epsilon}|_{\epsilon=0} \operatorname{dist}(b_i^{\epsilon c_1,\epsilon c_2}, S_i) < 0$ for $i = 1,2,3,4,5$, i.e. for small $\epsilon > 0$ the line $\epsilon c_1\ell_1 + \epsilon c_2\ell_{2345} + (1 - \epsilon c_1 - \epsilon c_2)\ell$ is closer to S_i than the Chebyshev line – a contradiction. □

REMARK 4. The use of the convex-concave property of the sections is almost unnecessary: interiors of any four parallel half-planes with pairwise non-parallel (see below) sides can be intersected by a line, which is as good as ℓ_{2345} for the proof.

3.4 Degenerate cases. In what follows we will always impose the following genericity assumption on H_i: we assume that ∂H_i are pairwise non-parallel (i.e. do not intersect in $\mathbb{R}P^3$).

For the degenerate cases (with some of the boundaries ∂H_i being parallel) the proof of existence of a good deformation is reduced via duality considerations to the Theorem 4. This reduction is constructed in details and in much more general situation in [KN1].

More exactly, let $B \subset \mathbb{R}^3$ be a convex-concave set, let L_α be a (finite) family of horizontal planes, and $S_\alpha = B \cap L_\alpha$ be corresponding sections. Choose in each L_α a half-plane $\widetilde{H}_\alpha$ supporting S_α, and suppose that the union of their boundaries $\bigcup \partial\widetilde{H}_\alpha$ intersects the infinite horizontal line in at most four points. In affine chart this means that there are at most four different directions such that boundary of each $\widetilde{H}_\alpha$ is parallel to one of these directions.

Theorem 6 (Three dimensional case of the last theorem in [KN1]). *There is a line intersecting all $\widetilde{H}_\alpha$.*

Here is how one can apply this result to the degenerate family of half-planes $H_1, \dots H_5$ constructed from a Chebyshev line as above. Let $\widetilde{H}_i$ be supporting to S_i half-planes with boundaries parallel to the corresponding H_i. The half-planes H_i contain the corresponding sections S_i in the interior by Lemma 7 (since the point of intersection of the Chebyshev line with the plane of S_i is the closest to S_i point of the boundary of H_i). Therefore $\widetilde{H}_i$ lie in the interiors of H_i, so the line whose existence is claimed in the theorem is a good deformation.

We briefly outline below results [KN1] leading to the Theorem 6.

The main fact is existence of a special L-duality preserving the class of

L-convex-concave sets satisfying the Main Conjecture 1. Here we describe this duality in three-dimensional case only.

First, suppose that the set $B \subset \mathbb{R}P^3$ is convex-concave with respect to a line $L \subset \mathbb{R}P^3$ and has a smooth boundary. The dual of this boundary (i.e. the set of all planes $\pi \subset \mathbb{R}P^3$ tangent to the boundary of B) divides the dual projective space $(\mathbb{R}P^3)^*$ into two parts. One of these parts contains the line L^* dual to L. The set $B_L^\perp \subset (\mathbb{R}P^3)^*$ L-dual to $B \subset \mathbb{R}P^3$ is the closure of the second part.

In general case, i.e. if the boundary of B is not smooth, another definition of the duality should be used. Namely, we define the set $B_L^\perp \subset (\mathbb{R}P^3)^*$ L-dual to $B \subset \mathbb{R}P^3$ as the union of all planes intersecting all sections of B by planes containing L.

We prove that this construction is a duality, i.e. that the dual set is again from the same class (i.e. convex-concave) and that dual of the dual of the set is the set itself.

Theorem 7 (see [KN1]). *Let $L^* \subset (\mathbb{R}P^3)^*$ be the line dual to the line $L \subset \mathbb{R}P^3$.*

- *The set $B_L^\perp \subset (\mathbb{R}P^3)^*$ is L^*-convex-concave.*
- *The set $B \subset \mathbb{R}P^3$ is L^*-dual to the set $B_L^\perp \subset (\mathbb{R}P^3)^*$.*

Remarkably, the Main Conjecture is true for both B and $B_L^\perp$ simultaneously: for any line $\ell \subset B$ the line ℓ^* dual to ℓ lies inside $B_L^\perp$ and vice versa. The proof of Theorem 6 consists of construction of a convex-concave set P whose L-dual contains a line by a simple application of the Theorem 4. By construction the sections of P by L_α are contained in $\widetilde{H}_\alpha$, so the dual of this line, as a line intersecting all these sections, is a required one.

Here we outline the proof of the Theorem 6.

The convex-concave set P mentioned above is a result of certain surgery on B. Namely, we replace each section of B by a convex polygon with $\leqslant 8$ sides circumscribed around the section. The sides of the polygons are tangent to the sections and parallel to the boundaries of $\widetilde{H}_\alpha$ (here we use the degeneracy condition to claim that we get at most octagons). It turns out that the resulting set P is convex-concave.

Consider the L-dual $P_L^\perp$ of P. Roughly speaking, sections of $P_L^\perp$ correspond to projections of P, and projections of $P_L^\perp$ correspond to sections of P. The set P has four remarkable projections, the projections along the directions of boundaries of $\widetilde{H}_\alpha$. Projections of P along other directions can be reconstructed from these four, since the sections are octagons. Similarly, the four sections S_i^*, $i = 1, 2, 3, 4$, of $P_L^\perp$ corresponding to these projections define $P_L^\perp$ in the following sense: the part of $P_L^\perp$ lying between two adjacent sections is the convex hull of these two sections.

Theorem 4 claims existence of a line ℓ intersecting all S_i. But any such line lies entirely inside $P_L^\perp$. The dual of this line lies inside P and therefore intersects all $P_i\alpha$. Since $P_\alpha \subset H_\alpha$, this line is a required one.

4 Combinatorial Properties of Half-planes Arising from a Chebyshev Line

In this and the next sections we investigate combinatorial properties of mutual position of the five half-planes constructed above. In this section we do not use the convex-concavity of the sections S_i (so the results are valid for any five convex compact figures lying on five horizontal planes in $\mathbb{R}^3$), and use only part of conditions implied by the fact that ℓ is the Chebyshev line for S_i. This, however, is enough to single out only six possible combinatorial types, and we deal with them one-by-one in the next sections.

There can be two types of good deformations – the ones intersecting ℓ (so-called trivial good deformations), and not intersecting ℓ. In this section we consider the restrictions on the mutual position of the half-planes implied by the absence of the good deformations of the first kind, i.e. the absence of lines intersecting ℓ and interiors of all half-planes H_i (recall that by the genericity assumption of §3.4 the boundaries of a half-plane are pairwise non-parallel).

We used an affine chart and a scalar product to define a Chebyshev line in the previous section, but the question we arrived at, i.e. the question if there is a line intersecting a given set of five half-planes, is a projective one. Here is a reformulation of the problem in projective terms. In $\mathbb{R}P^3$ we are given configurations consisting of

1. five different projective planes L_i, all containing the same line (further called a infinite horizontal line),
2. five half-planes $H_i \subset L_i$ – parts of these planes – containing convex (with respect to the infinite horizontal line) figures S_i together satisfying convex-concavity condition. The boundary of each half-plane consists of an infinite horizontal line and some other line. These other lines are pairwise nonintersecting (by the genericity assumption of §3.4);
3. a line ℓ intersecting all these other lines and not intersecting the

infinite horizontal line.

In this section we are looking for configurations without lines intersecting both ℓ and interiors of all half-planes. First we encode combinatorial properties of any configuration by a purely combinatorial code, leaving temporarily aside continuous parameters of the problem (double ratios of the intersections of the boundaries of half-planes with the infinite horizontal line and with the line ℓ) and the condition of absence of good deformations. This encoding can be done in several ways, so several codes correspond to each configuration. The condition of absence of trivial good deformations means that the configuration cannot be coded by a trivial code. In the next section we will see that there are at most six such configurations.

4.0.1 Coding. We will code combinatorial properties of configurations using projections from points $x \in \ell$ to the horizontal planes. As a result we will get a code — a permutation of numbers $1, 2, 3, 4, 5$ with signs.

The line ℓ is an affine part of a projective line $\widetilde{\ell} \cong \mathbb{RP}^1 \cong \mathbb{S}^1$. This projective line is divided into 5 intervals by its points of intersection with half-planes H_i. We choose a point $M \in \ell$ from one of these intervals and orientation on ℓ. We enumerate the points of the intersections of the half-planes with ℓ starting from M according to the chosen orientation, thus enumerating the half-planes H_i and planes L_i by numbers $1, 2, 3, 4, 5$.

Consider a projection $\pi : \mathbb{R}^3 \setminus L_M \to L_1$ with center M, where L_M is the horizontal plane passing through M. Take an orientation on the circle $\mathbb{S}^1 \subset L_1$ centered at $a_1 = \ell \cap L_1$ and a point $N \in \mathbb{S}^1 \setminus \cup \pi(\partial H_i)$. Thus we get an enumeration of the set of 10 points $\mathbb{S}^1 \cap (\cup_i \pi(\partial H_i))$ (recall that by the non-degeneracy assumption none of ∂H_i are parallel).

We can now write down a sequence of five numbers with signs (further called a code) which will encode the combinatorial properties of the configuration: on the i-th place of this sequence stands the number of the half-plane whose boundary projects onto the i-th point on $\mathbb{S}^1$ taken with $+$ if the projection contains the point N and with $-$ otherwise.

Remark 5. In the figures we denote the boundaries of $\pi(H_i)$ by their numbers i. The arrows point toward the projections of the corresponding half-planes.

4.0.2 Equivalent codes. In the coding procedure described above we made several choices. As a result we get several codes for the same situation. The resulting classes are in fact orbits of a group acting on the set of all possible codes.

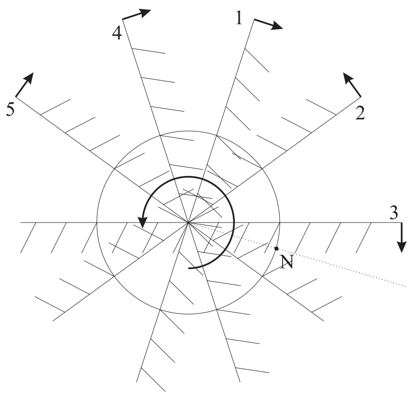

Figure 4: This projection and choices of N and of orientation of $\mathbb{S}^1$ correspond to the code $3+2-1+4+5+$

This group is generated by two pairs of generators. The first pair corresponds to the choices made on $\mathbb{S}^1$.

The first generator, denoted by β_1, corresponds to the moving of the point N to the previous interval. It acts on the code by cyclic permutation of the numbers and changing the sign of the last element: the i-th number goes to the $(i+1)$-th place except the first one which moves to the fifth place and changes sign, e.g. $\beta_1(1+2+3+4-5-) = 5+1+2+3+4-$.

The second generator, denoted by β_2, corresponds to the change of orientations on the circle. It acts on codes by symmetry: we should put the i-th number on the $5-i$-th place preserving the sign (e.g. $\beta_2(1{+}2{+}3{+}4{-}5{-}) = 5-4-3+2+1+$).

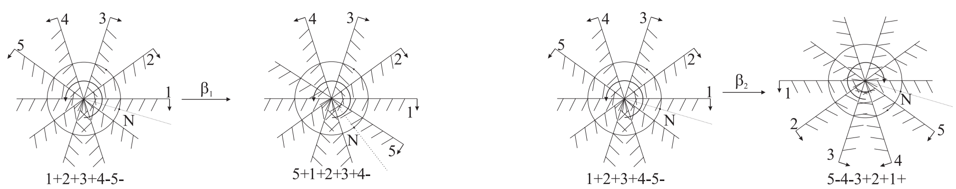

Figure 5: Action of β_1 and β_2

The second pair corresponds to the choices made on the Chebyshev line ℓ. In general, changing the position of the center of the projection or the orientation results not only in change of enumeration of half-planes but also in the different choice of the plane of projection. So in order to describe the effect of moving the point M to the next interval or changing the orientation of ℓ we have to identify somehow the planes of projections.

The third generator of the group, denoted by α_1, corresponds to the moving the point M to the point M' in the previous interval. If we identify planes L_1 and L_2 using the projections from M' and (upon this identification) make the same choice of N and of the orientation of $\mathbb{S}^1$, then α_1 acts on codes by changing 1+ to 2+ , 2+ to 3+, 3+ to 4+, 4+ to 5+, 5+ to $1-, \dots, 5-$ to 1+ (e.g. $\alpha_1(1+2+3+4-5- = 2+3+4+5-1+)$. In other words, the numeration shifts by 1 and the image of the fifth (from the M) half-plane flips.

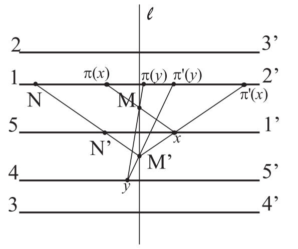

Figure 6: Projections from M and M' differ on H_5 and agree on H_1, H_2, H_3 and H_4

The fourth generator, denoted by α_2, corresponds to the change of orientation of ℓ. After identifying L_1 and L_5 by projection from M action of α_2 reduces to the renaming of the planes. So α_2 acts on codes by interchanging 5 with 1 and 4 with 2 with signs preserved (e.g. $\alpha_2(1-2+3+4-5-) = 5-4+3+2-1-)$. In other words, in the sequence of signs and numbers we rename numbers and leave signs intact.

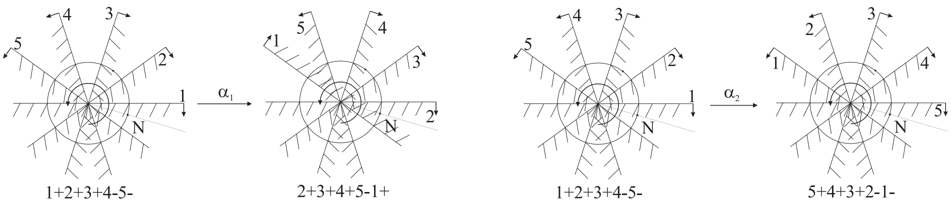

Figure 7: Action of α_1 and α_2

It is easy to see from this description that $\alpha_i\beta_j\alpha_i^{-1}\beta_j^{-1} = \alpha_1^{10} = \alpha_2^2 = \beta_1^{10} = \beta_2^2 = \mathrm{Id}$, $\alpha_2\alpha_1\alpha_2 = \alpha_1^{-1}$ and $\beta_2\beta_1\beta_2 = \beta_1^{-1}$, i.e. the group generated by α_i and β_i is $D_5 \bigoplus D_5$.

4.1 Cases of evident good deformation: trivial codes and the Chebyshev property. Here we translate the condition of absence of some good transformations mentioned in the beginning of the section to the language of codes.

4.1.1 Trivial codes. There are cases (i.e. combinatorial types of intersections of projections of H_i) which are forbidden for Chebyshev lines. These are in particular the cases when, for some choice of M, projections of all H_i have nontrivial intersection (i.e. more than one point). Indeed, in this case a good deformation which will intersect the Chebyshev line can be easily found.

Theorem 8. *A configuration corresponding to a Chebyshev line cannot be coded by a code containing* 1+, 2+, 3+ *and* 4+.

Proof. Suppose first that by choosing a point $M \in \ell$ and a point $N \in \mathbb{S}^1$ we get a code consisting of positive numbers only, i.e. a permutation of $1+, 2+, 3+, 4+$ and $5+$. By definition this means that the line connecting M and N intersects all H_i at their interior, i.e. is a good deformation.

If the code contains $5-$, then, after applying α_1^{-1}, we get an equivalent code with only positive entries, thus reducing to the previous case. □

4.1.2 Another easy case: the Chebyshev property. The following lemma uses for the first (and the last) time the Euclidean metric. More exact, it uses the definition of H_i as the set of all points $x \in L_i$ such that the angle $\widehat{xa_is_i}$ is acute (where $s_i \in S_i$ is the point of S_i closest to a_i). We will need this lemma only in the last section, where we consider the six nontrivial codes.

Theorem 9. *For a configuration constructed from a Chebyshev line no half-plane $H \subset L_1$ with $a_1 \in \partial H$ can contain S_1, $\pi(S_2)$, $\pi(S_3)$ and $\pi(S_4)$ simultaneously.*

Proof. Denote by N the endpoint of inward normal a_1N to ∂H.

We are given that $\pi(S_i) \subset H$ for $i = 1, 2, 3, 4$. Therefore $\pi(s_i) \in H$, so, by Lemma 8, $N \in \pi(H_i)$.

If $N \notin \pi(\partial H_i)$ then the code corresponding to N contains 1+, 2+, 3+ and 4+ and we are done by the previous lemma.

If not, we can move the point N slightly and get the same result. Namely, suppose that $N \in \pi(\partial H_i)$ for some i. Since (by the genericity assumption) no $\pi(\partial H_i)$ coincide, N cannot lie on more than one $\pi(\partial H_i)$. Therefore slightly moving N to the interior of this $\pi(H_i)$ we get a point N'

corresponding to a code containing $1+, 2+, 3+$ and $4+$, which is forbidden by Theorem 8. □

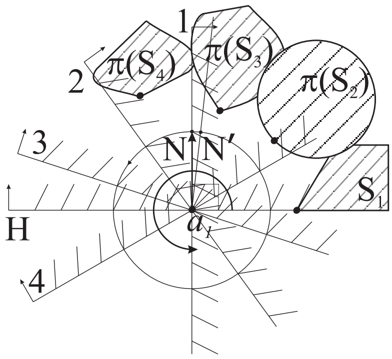

Figure 8: The case of $N \in \partial H_1$: the point N' lies in the interiors of H_1, H_2, H_3 and H_4

REMARK 6. This lemma generalizes the following simple geometrical fact:

LEMMA 11. *There is no half-space $H \subset \mathbb{R}^3$ with the Chebyshev line on its boundary containing all five sections S_i.*

Indeed, in this case in each plane L_i we will get a figure like in Lemma 8, so a line obtained from a Chebyshev line by a small parallel translation in the direction of the inward normal to ∂H will lie closer to all sections.

5 The Chess Board

In this section we single out all non-trivial codes, i.e. not equivalent to those named in Theorem 8. Though the number of codes is huge (namely $3840 = 2^5 5!$), there are only six equivalency classes not containing trivial codes. They are listed in the Theorem 10 below.

5.1 From a code to a corresponding chessboard. It is easier to visualize codes as a position of five rooks on a 5×5 chess board. This is done as follows: in the first column we put the rook in the row whose number is equal to the first number in the code. The color of the rook is white if this first number has sign + and black otherwise. We continue like this for the second, third, fourth and fifth columns (so if we forget the colors, the rook's position is exactly the graph of the permutation given by the code).

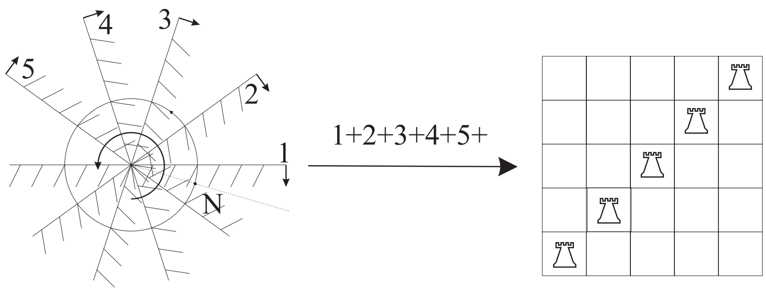

Figure 9: The code and the board corresponding to the projection above

It is easy to see that each column or row contains exactly one rook, i.e. the rooks do not threaten each other.

5.2 How the symmetry group acts on the rooks' positions. We described above an action of some symmetry group on codes. In the chess board realization the action of this group is remarkably simple:

- β_1 acts by moving the fifth column to the first place and changes the color of the rook standing in this column;
- α_1 acts in a similar way but with rows: α_1 moves the fifth row to the first place and changes the color of the rook standing in that row;
- β_2 acts by symmetry with respect to the vertical line;
- α_2 acts by symmetry with respect to the horizontal line.

5.3 Six equivalence classes consisting of nontrivial arrangements only. The trivial codes correspond to the arrangements of white rooks only, which will be called trivial arrangements. Our goal in this subsection is to exclude the rooks' arrangements equivalent to trivial ones.

Lemma 12. *Any arrangement not equivalent to a trivial one is equivalent to an arrangement with only one black rook. Moreover, this rook can be assumed not to stand on the border of the board.*

Proof. Pick any arrangement which is not equivalent to a trivial one. The β_1^5 simply changes all colors to the opposite ones, so we can assume that the number of black rooks is equal to one or two. The first case is what we need, so suppose that there are two black rooks. If one of them stands on the first or the last row, then using $\alpha_1^{\pm 1}$ we can change its color without changing the color of others, so leaving only one black rook. A similar statement holds for columns and β_1.

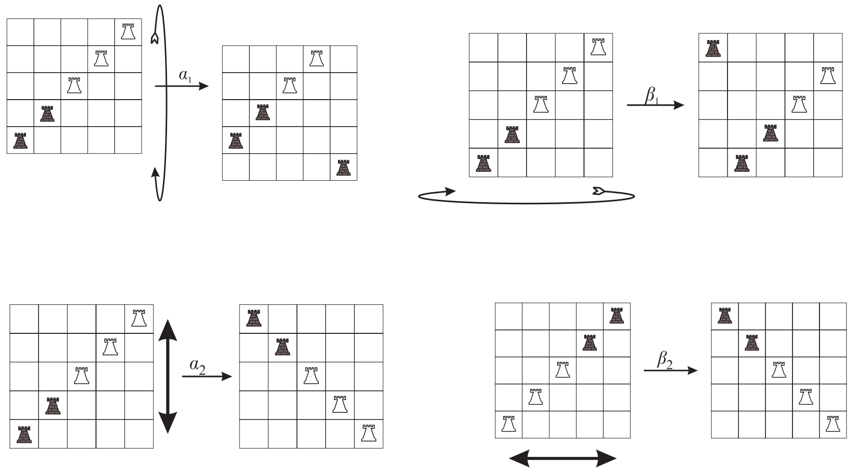

Figure 10: Action of generators of the group

So we can suppose that both black rooks are in the inner 3×3 square. Then we get at least two white rooks on the border. Take the fifth row. It contains one rook. Therefore the first or the fifth column should contain another white rook and moving this column and the fifth row (i.e. acting by $\beta_1\alpha_1$ or by $\beta_1^{-1}\alpha_1$) we arrive to a situation with four black rooks, which is equivalent (by β_1^5) to a situation with one rook only.

This black rook cannot stand on the border as otherwise by one move ($\alpha_1^{\pm 1}$ or $\beta_1^{\pm 1}$) we arrive at a trivial situation. $\square$

Using the symmetries α_2 and β_2, we can assume that a black rook occupies one of the four squares $(2,2), (2,3), (3,2), (3,3)$.

5.3.1 The case (2,2). Consider first the case of the black rook on the square $(2,2)$.

Lemma 13. *If one of the squares* $(1,1), (1,5), (5,1)$ *is occupied, then the position is trivial.*

Proof. Indeed, in these cases $\beta_1^{-1}\alpha_1^{-2}$ or $\beta_1^{-2}\alpha_1^{-1}$ or $\beta_1\alpha_1^{-2}$ correspondingly transforms the position to a trivial one. $\square$

Therefore in a position not equivalent to a trivial one, the white rook in the first column can occupy only one of the squares $(1,3)$ or $(1,4)$ and the square $(5,1)$ is empty.

5.3.2 White on (1,3) and Black on (2,2). This leaves four configurations:

C1 $3+2-1+4+5+$
C2 $3+2-1+5+4+$
C3 $3+2-4+1+5+$
C4 $3+2-5+1+4+$

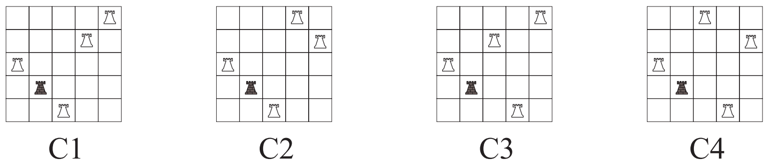

Figure 11: Four non-equivalent configurations C1–C4

5.3.3 White on (1,4) and Black on (2,2). There are another four possibilities (remember that $(5,1)$ is empty):

D2 $4+2-1+5+3+$
D3 $4+2-3+1+5+$
D4 $4+2-5+1+3+$
D5 $4+2-1+3+5+$

But D4 becomes trivial after $\alpha_1^{-3}\beta_1^2$, and D2 becomes D3 after $\beta_1^2\alpha_2\beta_2$. Moreover, after $\alpha_1^3\beta_1^{-3}$ C4 becomes D2. So the only new configuration is D5.

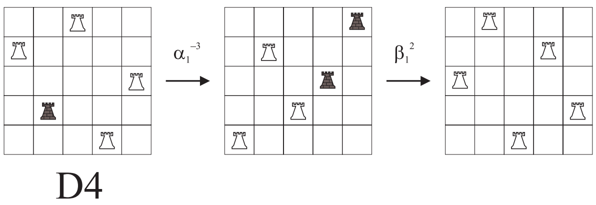

Figure 12: D4 becomes trivial after $\alpha_1^{-3}\beta_1^2$

5.3.4 Black on (2,3). Similarly to Lemma 13, the white rook in the first column cannot stand on the first or the last row. In other words, in a position with a black rook on $(2,3)$ and not equivalent to a trivial one, the squares $(1,5)$ and $(1,1)$ are empty. Indeed, $\alpha_1^{\pm 1}\beta_1^{-2}$ correspondingly trivialize these arrangements.

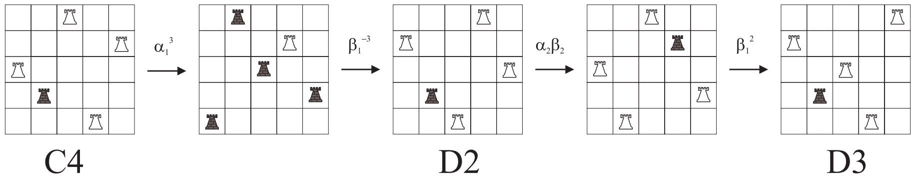

Figure 13: C4 is equivalent to D2 and to D3

So the only places the white rook can stand are $(1,2)$ or $(1,4)$. These positions are in fact equivalent by α_2, so we can consider the positions with a white rook on $(1,4)$ and the black rook on $(2,3)$.

But these positions are equivalent by $\alpha_2\beta_2\beta_1^{-2}$ to the positions with a black rook on $(2,2)$, so are in fact considered above.

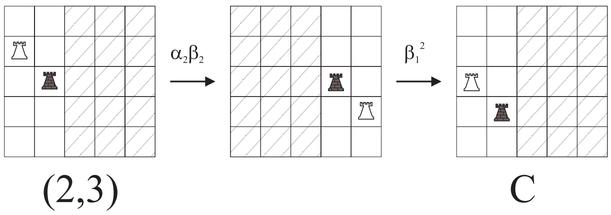

Figure 14: The series of cases of the black rook on $(2,3)$ is equivalent to the series C

5.3.5 Black on (3,2). These arrangements are also equivalent to arrangements with the (only) black rook on $(2,2)$. The proof repeats word-for-word the proof above changing β to α and α to β everywhere. This is because the actions of the group is symmetric with respect to the diagonal (though this symmetry isn't itself in the group).

5.3.6 Black rook on (3,3). The complement of the square to the third row and the third column consists of four two-by-two squares.

LEMMA 14. *If the arrangement is not equivalent to a trivial one, then each square contains exactly one rook.*

Proof. Indeed, if not, then one of them contains two rooks and the opposite should necessarily contain the other two (since exactly one rook stands in each row and in each column). Applying β_2 if necessary, one can assume that these are the lower left and the upper right squares. Then $\alpha_1^2\beta_1^3$ transforms the arrangement to a trivial one. □

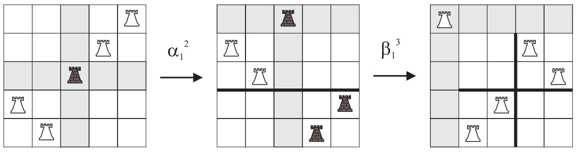

Figure 15: Triviality of the case of the black rook on $(3,3)$ and one of the squares containing two rooks

Lemma 15. *If one of rooks stands in the corner* (*i.e. on* $(1,1),(1,5),(5,1)$ *or* $(5,5)$), *then the situation is equivalent to a situation with the only black rook standing on* $(2,2)$ (*i.e. is in fact considered above*).

Proof. Using α_2 and β_2, if necessary, we can suppose that the white rook stands on $(1,1)$. Then we get a situation with the only black rook on $(2,2)$ after $\alpha_1^{-1}\beta_1^{-1}$. □

Corollary 2. *All configurations with one white rook in the inner* 3×3 *square are trivial or have a rook in a corner.*

Proof. Suppose that $(2,2)$ is occupied and the position is neither trivial nor with a rook in a corner. Then the 2×2 square contain one rook each. Then the squares $(1,4)$ and $(4,1)$ are occupied, since the corners are empty and the second row and second column already contain a rook. Therefore the only remaining square for the fourth rook is in the corner $(5,5)$, which is forbidden. □

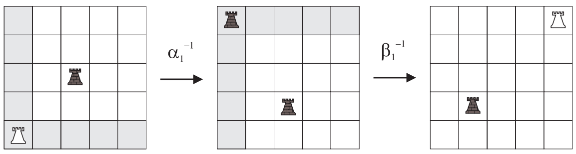

Figure 16: The case of the black rook on $(3,3)$ and one of the rooks in the corner is equivalent to the series C

The only remaining positions are $4+1+3-5+2+$ and $2+5+3-1+4+$, which are equivalent by α_2 or β_2.

5.3.7 The final list.

It consists of six variants.

Theorem 10. *A configuration corresponding to a Chebyshev line should be equivalent to a configuration described by one of the following codes*

C1 $3+2-1+4+5+$ C2 $3+2-1+5+4+$ C3 $3+2-4+1+5+$

C4 $3+2-5+1+4+$ D5 $4+2-1+3+5+$ E6 $4+1+3-5+2+$

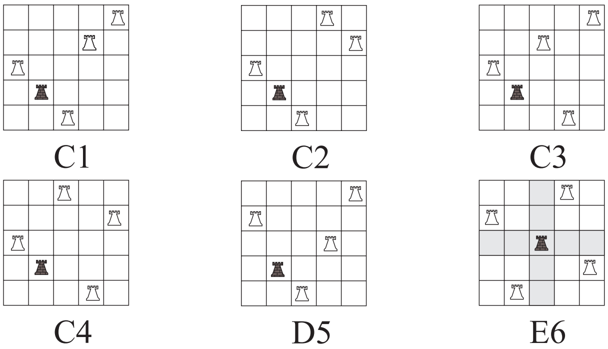

Figure 17: The six nontrivial variants

6 Non-triviality of a Code and Convex-concavity Imply Existence of a Good Deformation

In this section we consider the six nontrivial cases of Theorem 10. Each case has several continuous parameters (e.g. angles between ∂H_i, distances between L_i), and only for some choice of parameters the configuration of half-planes arises from a Chebyshev line. In other words, for only part of the parameter space parameterizing this combinatorial type, the corresponding configuration of half-planes does not admit a good deformation. Indeed, Theorem 10 excludes only codes admitting a good deformation *intersecting the Chebyshev line* ℓ, and do not deal with good deformations not intersecting ℓ.

In what follows we show that the configurations of half-planes arising from sections S_i of a convex-concave body all admit a good deformation. Therefore they cannot correspond to a Chebyshev line, so the assumption that the Chebyshev line doesn't intersect the sections leads to a contradiction.

More exactly, we extract from the convex-concavity condition some inequality between the double ratio of angles between ∂H_i and the double ratios of distances between L_i in some particular combinatorial assumptions. This inequality implies existence of a line intersecting four of the half-planes H_i in some particular sectors. For five of the six cases of Theorem 10 these assumptions are satisfied, and moreover the resulting line automatically intersects the fifth half-plane. The sixth case E6 simply cannot occur for convex-concave sections.

The main tool in the proofs is Theorem 4, only applied now to some parts of the sections S_i. The only Euclidean property we will need is Theorem 9, whose statement is projective. So we can move the center of projection to infinity, and the projection becomes a parallel projection $\pi : \mathbb{R}^3 \to L_1$ along the z-axis, with S_i are ordered by their z-coordinate.

We will also use a linear structure defined on L_1 defined by the coordinates x and y (i.e. we take the point a_1 as the origin).

6.1 Sectorial Browder theorem. We will denote by $\pi(H_i)^c$ for the closure of $L_1 \setminus \pi(H_i)$. We define half-spaces $B_i = \pi^{-1}(\pi(H_i))$ and denote by B_i^c the closure of their complements.

Theorem 11. *Suppose that*

1. $H_1 \cap \pi(H_4)^c \subset \pi(H_2)^c$ *and*
2. $\pi(H_3) \cap \pi(H_2) \subset \pi(H_4)^c$.

Suppose moreover that $S_1 \cap \pi(H_4)^c \neq \emptyset$. *Then* $\pi(S_2) \cap \pi(H_3)$, $\pi(S_3) \cap \pi(H_2)$ *and* $\pi(S_4) \cap H_1^c$ *are also non-empty and there exists a straight line* L *intersecting* $S_1 \cap B_4^c$, $S_2 \cap B_3$, $S_3 \cap B_2$ *and* $S_4 \cap B_1^c$.

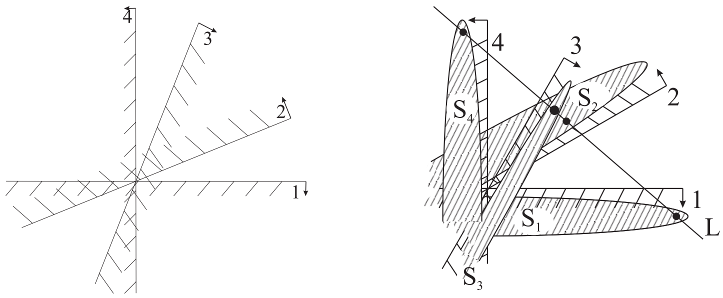

Figure 18: The configuration of half-planes and projection of the line L from Theorem 11

In our notation the conditions (1) and (2) mean existence of the subsequence 1+2−3+4− in a sequence coding the configuration. In applications below the condition $S_1 \cap \pi(H_4)^c \neq \emptyset$ will follow from Lemma 20 below.

Proof. First we prove two combinatorial lemmas:

Lemma 16. $H_1 \cap \pi(H_4)^c \subset \pi(H_3)$.

Proof. Suppose that $H_1 \cap \pi(H_4)^c \not\subset \pi(H_3)$. Since boundaries of the half-planes are pairwise different, there is a point x lying in the interior of

$(H_1 \cap \pi(H_4)^c) \setminus \pi(H_3)$. Then $-x \in H_1^c \cap \pi(H_4) \cap \pi(H_3) \subset \pi(H_2) \cap \pi(H_3)$ by assumption and also $-x \in \pi(H_4)$ – a contradiction. □

LEMMA 17. $\pi(H_2) \cap \pi(H_3) \subset \pi(H_1)^c$.

Proof. As before, take x in the interior of $\pi(H_2) \cap \pi(H_3) \cap H_1$. Then $x \in \pi(H_4)^c \cap \pi(H_1)$ by the assumption (2) and therefore $x \in \pi(H_2)^c$ by the assumption (1) – contradiction. □

Our claim will be proved by applying Theorem 4 to $S_1 \cap \pi(H_4)^c$ as B, $S_2 \cap B_3$ as A, $S_3 \cap B_2$ as C and $S_4 \cap B_1^c$ as D. Let's check conditions of Theorem 5. In other words, we have to check that

1. a line passing through $S_1 \cap B_4^c$ and intersecting S_2 and S_3 (existing by convex-concavity) intersects $S_2 \cap B_3$ and $S_3 \cap B_2$, and
2. a line passing through $S_3 \cap B_2$ and intersecting S_1 and S_4 (existing by convex-concavity) intersects $S_4 \cap B_1^c$ and $S_1 \cap B_4^c$.

(Clearly $S_1 \cap B_4^c$, $S_2 \cap B_3$, $S_3 \cap B_2$ and $S_4 \cap B_1^c$ are compact and convex).

Let a line intersect $S_1 \cap B_4^c$ and S_2 and S_3 at points c_1, c_2 and c_3 accordingly. Necessarily c_2 lies between c_1 and c_3. We know that $c_1 \in S_1 \cap \pi(H_4)^c \subset H_1 \cap \pi(H_4)^c \subset \pi(H_2)^c \cap \pi(H_3)$. Since $c_1, c_3 \in B_3$, so $c_2 \in B_3$ (so $S_2 \cap B_3$ is non-empty). Similarly, $c_1 \in B_2^c$ and $c_2 \in B_2$, so $c_3 \in B_2$ (and $S_3 \cap B_2$ is non-empty). So the first claim is proved.

Similarly, let a line intersect $S_3 \cap B_2$ and S_4 and S_1 at points c_3, c_4 and c_1 accordingly. As before, $S_3 \subset B_4^c \cap B_1^c$. Since $c_4 \in B_4$ and $c_3 \in B_4^c$, so $c_1 \in S_1 \cap \pi(H_4)^c$. Since $c_3 \in B_1^c$, so $c_4 \in S_4 \cap B_1^c$ (so in particular $S_4 \cap B_1^c$ is not empty). The second claim follows. □

6.2 Double ratios. After projecting a configuration satisfying conditions of Theorem 11 to the plane L_1 we obtain a figure below.

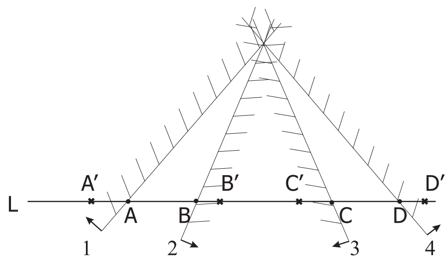

Figure 19: Double ratios of planes L_i and projections of boundaries of H_i

Here L is the projection of the line existing by Theorem 11. By

A', B', C', D' we denote intersections of L with S_i and by A, B, C, D intersections of L with $\partial(\pi(H_i))$.

The existence of the line L implies an inequality between the double ratio of distances between L_i and the double ratio of directions of boundaries of H_i. Namely, denote the double ratio $AB/BD : AC/CD$ of points A, B, C, D by (A, B, C, D). Then $(A'B'C'D')$ is exactly the double ratio of distances between L_i:

$$(A', B', C', D') = \frac{h_1}{h_1 + h_2} : \frac{h_1 + h_2}{h_3},$$

where h_i is the distance between L_i and L_{i+1}. (A, B, C, D) is the double ratio of directions of ∂H_i, and the following inequality holds:

Corollary 3. *In the assumptions of Theorem 11 the double ratio of distances between L_i is strictly smaller than the double ratio of directions of ∂H_i:*

$$(A', B', C', D') > (A, B, C, D).$$

Proof. Indeed, the configuration of the points A', B', C', D' is obtained from the points $ABCD$ by movements which only increase the above double ratio:

1. $(A, B, C, D) < (A', B, C, D)$ since $A'B/A'C > AB/AC$,
2. $(A', B, C, D) < (A', B', C, D)$ since $A'B'/B'D > A'B/BD$,
3. $(A', B', C, D) < (A', B', C', D)$ since $C'D/A'C' > CD/A'C$,
4. $(A', B', C', D') < (A', B', C', D)$ since $C'D'/B'D' > C'D/B'D$.

The equality is possible only if all points A', B', C', D' lie on the corresponding lines, which is impossible since, for example, the point B' lies in $\pi(S_2)$ which is included in the *interior* of the half-plane $\pi(H_2)$, so $B' \neq B$ and the inequality in (2) is strict. □

Lemma 18. *With conditions as above suppose that four points A'', B'', C'' and D'' lie on a line L' and*

- $A'' \in \partial H_1 \setminus \pi(H_4)$
- $C'' \in \partial(\pi(H_3)) \setminus \pi(H_4)$
- $D'' \in \partial(\pi(H_4)) \setminus H_1$

Suppose moreover, that $A''B'' : B''C'' : C''D'' = A'B' : B'C' : C'D'$. Then B'' lies in the interior of $\pi(H_2) \cap \pi(H_3)$.

Proof. This follows directly from the inequality Corollary 3. Indeed, let $\overline{B} = L' \cap \partial(\pi(H_2))$. Then

$$(A'', \overline{B}, C'', D'') = (A, B, C, D) < (A', B', C', D') = (A'', B'', C'', D'').$$

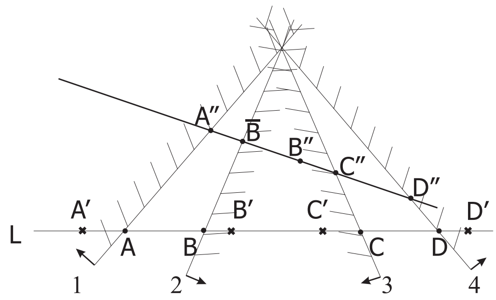

Figure 20: Line L' of Lemma 18

This is equivalent to $A''\overline{B}/\overline{B}D'' < A''B''/B''D''$, which is possible only if B" is between $\overline{B}$ and D'', i.e. $B'' \in \pi(H_2)$. Since $B'' \in [A''C'']$, also $B'' \in \pi(H_3)$. □

The lemma means that the line, whose existence is claimed in Theorem 11, can be moved in such a way that it will still intersect the interior of H_2 and will also intersect boundaries of H_1, H_3 and H_4.

6.3 The six non-trivial configurations which contradict convex-concavity. We will call a stencil any five points $c_1, c_2, c_3, c_4, c_5 \in L_1$ which are projections points of intersections of some line $l \subset \mathbb{R}P^3$ with L_i, $c_i = l \cap L_i$. Note that $|c_1c_2| : |c_2c_3| : |c_3c_4| : |c_4c_5|$ is the same for all stencils and is equal to $h_1 : h_2 : h_3 : h_4$ where h_i are the distances between L_i and L_{i+1}. Evidently this is a necessary and sufficient condition for five points in L_1 lying on a line in this order to be a projection of points of intersection of some line in $\mathbb{R}P^3$ with the planes L_i.

A projection of a good deformation is a stencil with an additional property $c_i \in \pi(H_i)$, with at least one c_i lying in the interior of $\pi(H_i)$. Vice versa, any such stencil is a projection of a good deformation.

We can reformulate Lemma 18 using this notation:

LEMMA 19. *Suppose that*

1. $H_1 \cap \pi(H_4)^c \subset \pi(H_2)^c$,
2. $\pi(H_3) \cap \pi(H_2) \subset \pi(H_4)^c$ *and*
3. $S_1 \cap \pi(H_4)^c \neq \emptyset$.

Then there exists a stencil such that

1. $c_1 \in \partial H_1 \cap \pi(H_4)^c$,
2. c_2 *lies in the interior of* $\pi(H_2) \cap \pi(H_3)$,
3. $c_3 \in \partial\pi(H_3) \cap \pi(H_4)^c$ *and*

4. $c_4 \in \partial\pi(H_4) \cap \pi(H_1)^c$.

Similar statements hold for all strictly increasing subsequence of 12345 consisting of four numbers (i.e. 1245 or 1345, etc. instead of 1234).

6.3.1 Chebyshev property. Here we prove that one of the consequences of the Chebyshev property formulated in Lemma 9 is that the set $S_1 \cap \pi(H_4)^c$ in Lemma 18 is never empty.

LEMMA 20. *If $S_1 \subset \pi(H_4)$ or $S_1 \subset \pi(H_5)$ then the configuration is trivial.*

Proof. In the first case $\pi(S_2)$ and $\pi(S_3)$ also lie in $\pi(H_4)$ by convex-concavity. Indeed, any point of S_2 lies on a segment with endpoints on S_1 and S_4, and projection of such a segment lies entirely in $\pi(H_4)$. The same is true for S_3, so by Lemma 9 the configuration is trivial. Similarly, in the second case $S_i \subset \pi(H_5)$ for $i = 1, 2, 3, 4, 5$ and again by Lemma 9 the configuration is trivial. □

In cases C1, C3, C4 and D5, Lemma 19 and Lemma 20 immediately give the existence of a stencil which is a projection of a good deformation.

6.3.2 The C1 case. This is the case $3 + 2 - 1 + 4 + 5+$. We will consider an equivalent (after $\beta_1^{-2}\beta_2$) variant $1 + 2 - 3 + 5 - 4-$.

If $S_1 \cap \pi(H_4)^c = \emptyset$ then the configuration is trivial by Lemma 20. So $S_1 \cap \pi(H_4)^c \neq \emptyset$ and Lemma 19 is applicable to the subsequence $1 + 2 - 3 + 4-$ of the code.

In the resulting stencil $c_4 \in \pi(H_5)$ and $c_1 \notin \pi(H_5)$. Indeed, the sector $H_1 \cap \pi(H_4)^c$ is the smallest sector bounded by boundaries of half-planes and containing the point N. Since $N \notin \pi(H_5)$, so $H_1 \cap \pi(H_4)^c \cap \pi(H_5) = \emptyset$. This means that $-c_4, c_1 \notin \pi(H_5)$.

Therefore the point c_5 of the stencil lies in $\pi(H_5)$. Therefore the line projecting to this stencil is a good deformation.

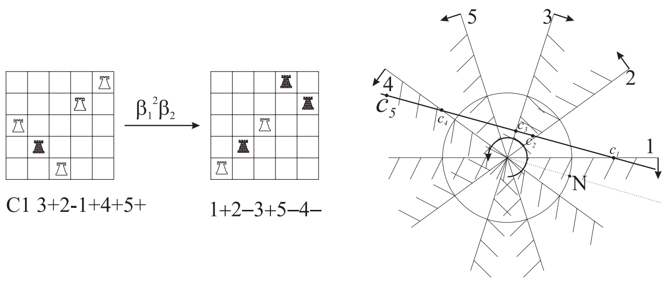

Figure 21: The case C1

6.3.3 The C3 case. This is the case of $3+2-4+1+5+$. We will consider the equivalent (after $\beta_2\alpha_1^6$) case of $1+2-5-3+4-$.

As above, $S_1 \not\subset \pi(H_4)$ by Lemma 20, so we can apply Lemma 19 to the subsequence $1+2-3+4-$ of the code, exactly as in the case C1. As before, c_1 lies on $\partial H_1 \cap \pi(H_4)^c$ and therefore in $\pi(H_5)^c$. Also, $c_4 \in \partial\pi(H_4) \cap H_1^c$ and therefore $c_4 \in \pi(H_5)$. So c_5 also lies in $\pi(H_5)$ since c_5 and c_1 lie on different sides of c_4. Therefore the stencil given by Lemma 19 is a projection of a good deformation.

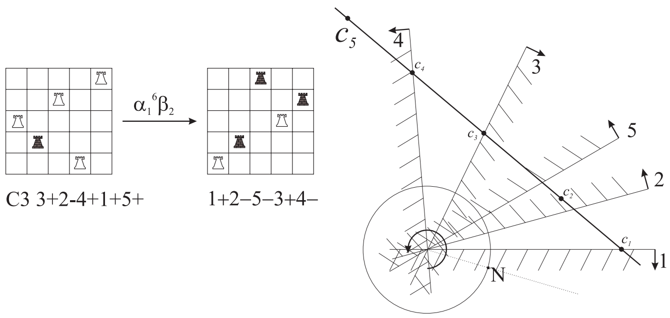

Figure 22: The case C3

6.3.4 The C4 case. This is the case of $3+2-5+1+4+$. We will consider the equivalent (after $\beta_1^2\beta_2\alpha_1^{-1}$) case of $1+2-3+5-4+$.

As before, by Lemma 20, $S_1 \not\subset B_5$. We apply Lemma 19 to the subsequence $1+2-3+5-$ and get a stencil with $c_1 \in \partial H_1 \cap \pi(H_5)^c$, $c_2 \in \pi(H_2)$, $c_3 \in \partial\pi(H_3) \cap \pi(H_5)^c$ and $c_5 \in \partial\pi(H_5) \cap H_1^c$. Then $c_4 \in \pi(H_4)$. Indeed, $c_1, c_5 \in \pi(H_4)$ and c_4 lies between c_1 and c_5. So this stencil is a projection of a good deformation.

6.3.5 The D5 case. This is the case of $4+2-1+3+5+$. It is equivalent (after $a_1^6 b_2$) to the case $1+4-2-3+5-$.

Again, $S_1 \not\subset B_5$ by Lemma 20. We apply Lemma 19 to the subsequence $1+2-3+5-$ and get a stencil with $c_1 \in \partial H_1 \cap \pi(H_5)^c$, $c_2 \in \pi(H_2)$, $c_3 \in \partial\pi(H_3) \cap \pi(H_5)^c$ and $c_5 \in \partial\pi(H_5) \cap H_1^c$. Now $c_4 \in \pi(H_4)$ follows from the fact that $c_3, c_5 \in \pi(H_4)$ and c_4 lies between c_3 and c_5. So this stencil is a projection of a good deformation.

In the two last cases we should exhibit a little more inventiveness.

The case C2 requires double application of Lemma 19, whereas in E6 the combinatorial properties of the intersections contradict Theorem 11.

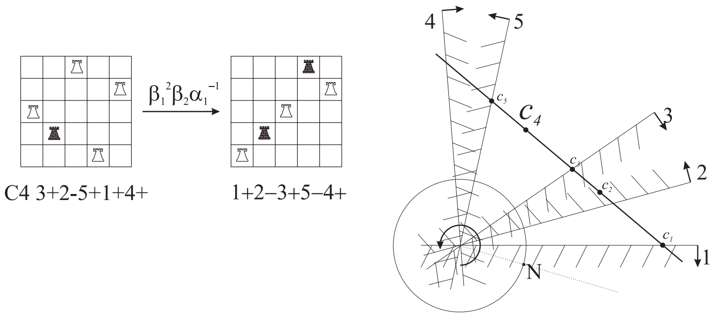

Figure 23: The case C4

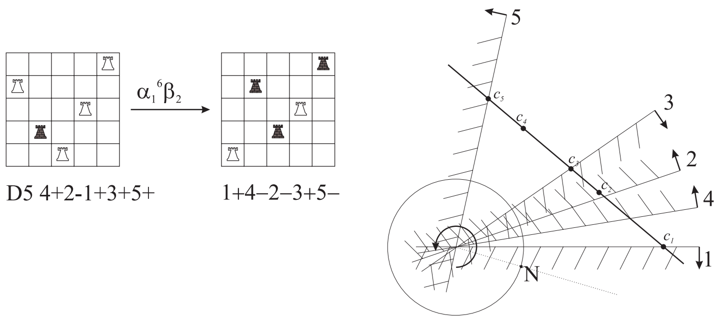

Figure 24: The case D5

6.3.6 The C2 case. This is the case of $3+2-1+5+4+$. After applying $\alpha_1^3\beta_1\beta_2$ it will transform into an equivalent variant 1+2−3−4+5−.

By Lemma 20 $S_1 \cap \pi(H_5)^c \neq \emptyset$. Applying Lemma 19 to the sequences $1+2-4+5-$ and $1+3-4+5-$ we see that there are two stencils, one with points $c_1c_2c_3c_4c_5$ and another with points $c_1'c_2'c_3'c_4'c_5'$, such that the following conditions hold

1. $c_1, c_1' \in \partial H_1 \cap \pi(H_5)^c$,
2. $c_2 \in \pi(H_2) \cap \pi(H_4)$,
3. $c_3' \in \pi(H_3) \cap \pi(H_4)$,
4. $c_4, c_4' \in \partial\pi(H_4) \cap \pi(H_5)^c$ and
5. $c_5, c_5' \in \partial\pi(H_5) \cap \pi(H_1)^c$.

But any two stencils satisfying (1), (4) and (5) differ only by a dilatation centered at the origin and these dilatations preserve $\pi(H_i)$. So $c_3 \in \pi(H_3) \cap \pi(H_4)$ and we get the stencil which is a projection of a good deformation.

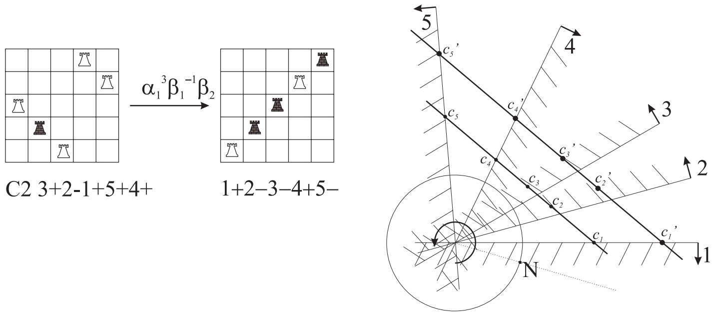

Figure 25: The case C2

6.3.7 The E6 case. This is the case of $4+1+3-5+2+$. It is equivalent (by β_1^4) to $1-3+5-2-4+$. Recall that $B_i = \pi^{-1}(H_i)$.

Suppose first that $S_1 \cap \pi(H_3) \neq \emptyset$. Similarly to the proof of Theorem 11, we will apply Theorem 4 to $S_1 \cap \pi(H_3)$ as B, $S_4 \cap B_2$ as C and S_2 and S_3 as A and D correspondingly and will arrive at a contradiction.

Construct two mappings, $h_1 : CSet(S_1 \cap B_3) \to CSet(S_4 \cap B_2)$ and $h_2 : CSet(S_4 \cap B_2) \to CSet(S_1 \cap B_3)$, as in the proof of Theorem 4. Namely, take a point $b \in S_1 \cap B_3$. There is a line passing through this point and section S_2 and intersecting the section S_4 at point c. Since $\pi(b) \in \pi(S_1) \cap \pi(H_3) \subset \pi(H_2)^c$ and evidently $\pi(S_2) \subset \pi(H_2)$, we conclude that $\pi(c) \in \pi(S_4) \cap \pi(H_2)$, i.e. $c \in S_4 \cap \pi(H_2)$. The mapping h_1 is the extension to the closed subsets of $S_1 \cap B_3$ of the mapping sending the points a to the set of all such c. Similarly, to define h_2 take any point $c \in S_4 \cap B_2$. There is a line passing through this point and intersecting the section S_3 and the section S_1 at a point a. Since $\pi(c) \subset \pi(H_4) \cap \pi(H_2) \subset \pi(H_3)^c$ and $\pi(S_3) \subset \pi(H_3)$, we get that $a \in S_1 \cap B_3$.

By virtue of Theorem 4 this proves existence of a line intersecting $S_1 \cap \pi(H_3)$, S_2, S_3 and $S_4 \cap B_2$.

But this line cannot exist. Indeed, denoting the projections of the intersection points by $c_1c_2c_3c_4$ we see that $c_2, c_4 \in \pi(H_4) \cap \pi(H_2) \subset \pi(H_3)^c$ and therefore the point c_3 – lying between c_2 and c_4 – should also belong to B_3^c, which contradicts $c_3 \in \pi(H_3)$.

Therefore $S_1 \subset \pi(H_3)^c$. By convex-concavity we get that $\pi(S_4), \pi(S_5) \subset \pi(H_3)$ (any point of these sections is an endpoint of a segment intersecting S_3 with another endpoint in S_1). Therefore $\pi(S_5) \subset \pi(H_5) \cap \pi(H_3) \subset \pi(H_2)$ and $\pi(S_4) \subset \pi(H_4) \cap \pi(H_3) \subset \pi(H_2)^c$.

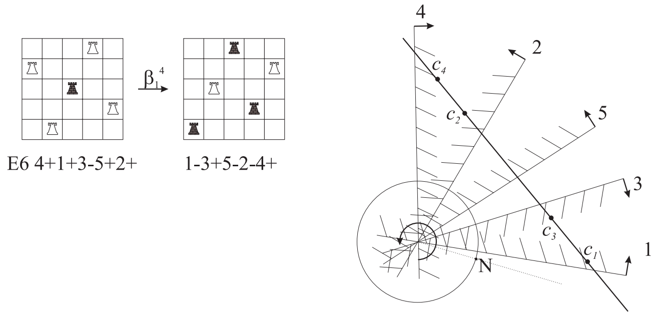

Figure 26: The case of $S_1 \cap \pi(H_3) \neq \emptyset$ is impossible

This is incompatible with the existence of lines joining S_5, S_4 and S_2 given by the convex-concavity condition. Indeed, take any segment intersecting S_2, S_4 and S_5 at points s_2, s_4 and s_5 correspondingly. Its projection $[\pi(s_2), \pi(s_5)]$ has both ends in $\pi(H_2)$, so $\pi(s_4) \in \pi(H_2)$ as well, which contradicts $\pi(s_4) \in \pi(S_4) \subset \pi(H_2)^c$.

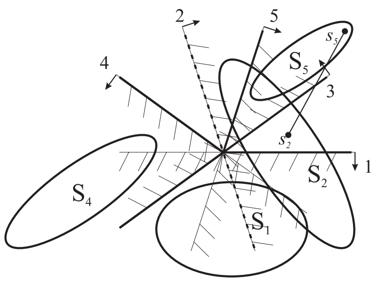

Figure 27: The case of $S_1 \subset B_3^c$ is impossible (the previous picture is rotated by 180°)

End of the proof of Theorem 1. Existence of the Chebyshev line implies by Theorem 10 that one of the combinatorial cases C1–C4, D5 or E6 should occur. We just proved that in each of these cases the convex-concavity condition is incompatible with the existence of the Chebyshev line: either a good deformation exists (C1–C4, D5), or the case cannot occur for the sections of the convex-concave set (E6). So the assumption of the existence of the Chebyshev line led us to a contradiction.

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