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BRANCHED COVERS OF S^2 AND BRAID GROUPS

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We study branched covers of S^2, i.e. meromorphic functions on Riemann surfaces, up to topological equivalence. Luroth (1871) and Clebsch (1873) proved that the topological type of a branched cover of S^2 in generic position is determined by the number of branch points (critical values) and the number of leaves of the cover. We give some conditions for the local characteristics around the critical values of a nongeneric branched cover to determine its topological type.

Specifically, (1) in the case of polynomial maps $S^2 \to S^2$ we prove that the topological type of the cover is uniquely determined by its local characteristics not only in the generic case, but also for all degenerate maps of sufficiently small codimension. Namely, we prove the uniqueness of the topological type for all degeneracies up to codimension about $k/4$, where k is the degree of the polynomial, and give examples to show the nonuniqueness in codimension about $k/2$.

(2) We give an elementary combinatorial proof showing that the topological type of a map with only one degenerate critical value is determined by its local branching characteristics. A topological proof of this was given by Natanzon (1988).

(3) We give the topological classification of polynomials of degree ≤ 6 and compute the number of topological classes of some special 3- and 4-fold covers of S^2.

(4) We prove the statement of Thom (1965) that each set of local characteristics satisfying some simple (necessary) conditions actually does correspond to a polynomial.

Some of our proofs are based on the action of the braid group on the set of branched covers of S^2. The reduction of the topological classification of branched covers of S^2 to the orbits of this action was described in [22]. In the actual calculation of the topological classes of polynomial maps, the "pictures" considered in [22] are helpful. For the special case of polynomial mappings with three critical values, these pictures reduce to the dessins d'enfant introduced later by Grothendieck in the more general situation of algebraic maps with three critical values.

Note that some conditions on the local branching characteristics guaranteeing the topological uniqueness of the branched cover were announced by Protopopov (1988). However, for polynomial maps, Protopopov's statements give only Luroth-Clebsch's result.

1. Introduction

Let $f : M \to S^2$ be a k-fold cover of the 2-sphere S^2 by a surface M of genus $g \geq 0$ that is branched over n points $b_1, \ldots, b_n$ (i.e., $f^{-1}(b_i)$, $i = 1, \ldots, n$, consists of $< k$ points; the b_i's will be called the *critical values* of f).

Two such covers f_1 and f_2 are said to be *T-equivalent* ("T" for "Thom" and "topological") if there exist orientation-preserving homeomorphisms $g : M \to M$ and $h : S^2 \to S^2$ such that

$$f_2 \circ g = h \circ f_1.$$

These T-equivalence classes of maps were considered by Thom [20] in his solution of the problem of the existence of a complex polynomial with prescribed critical values (the analogous real problem was posed and solved by Davis [9]). Thom also asked for a T-classification of such covers of S^2.

Consider the set $N \subset S^2$ consisting of n-th roots of 1. It is easy to see that each cover of S^2 that is branched over n points is T-equivalent to a cover that is branched over N.

Take N as base point in the space $\mathcal{C}^n$ of unordered n-tuples of points in $S^2 \setminus \infty$, and identify the plane $\mathbb{C}$ equipped with the segment $[1, n]$ with $S^2 \setminus \infty$ equipped with $e^{2\pi i t/n}$, $t \in [1, n]$. This determines an isomorphism of the Artin braid group $B(n)$ with the fundamental group of $\mathcal{C}^n$ based at N, and an action of $B(n)$ on the space of covers over S^2 that are branched over N.

The following Theorem is contained in [22]:

Theorem. *There is a one-to-one correspondence between the set of T-equivalence classes of covers of S^2 branched over n points and the orbits of the action of the Artin braid group $B(n)$ on the set of covers of S^2 that are branched over N.*

Note that the number k of sheets of the cover and the number n of critical values are obvious invariants under T-equivalence.

Further, to each critical value b there corresponds a partition of k into summands d_i (equal to the orders of the points of $f^{-1}(b)$; if $x_i \in f^{-1}(b)$ has order $d_i > 1$, then x_i is called a *critical point* and $d_i - 1$ is its *multiplicity*). Call this partition the *type* of the critical value. We thus get a set Σ of n partitions of k. This set Σ of local data is also a T-invariant, which we call (following Protopopov [18]) the *passport* of the cover.

The set of topological classes (up to homeomorphism of the domain) of covers of S^2 branched over N and with given passport is finite and can be computed explicitly. The action of the $n-1$ standard generators of $B(n)$ on this set also can be computed explicitly. Thus, we can explicitly enumerate all the T-equivalence classes of covers of S^2 branched over N.

In the actual calculations of T-equivalence classes of polynomials, the "pictures" that were considered in [22] are helpful. Let $P : S^2 \to S^2$ be a polynomial in one complex variable such that the set of finite critical values of P is the set N of nth roots of 1. The *picture* of P is the inverse image under P of the unit disc.

The purpose of this paper is to investigate the following

Problem. *Give conditions on a set Σ of partitions of k that imply that there is a unique T-equivalence class of branched covers of S^2 with passport Σ.*

Here are some earlier results of this kind.

Luroth [15] and Clebsch [8] showed that if all the critical values of $f : M \to S^2$ are simple (a critical value $b \in S^2$ of a k-fold branched cover $f : M \to S^2$ is said to be *simple* if $f^{-1}(b)$ consists of $k-1$ points), then f is determined up to T-equivalence by the number k of sheets of f and the number n of critical values.

Natanzon [17] extended this result to the case in which all but possibly one of the critical values of f are simple, by using the topology of surfaces (cut-and-paste arguments).

Other authors (see, for example, [5], [10]) have extended Clebsch's theorem to the case of branched covers $f : M_1 \to M_2$ between two surfaces with only simple critical values.

In Sec. 2 we shall give an elementary combinatorial proof of Natanzon's theorem.

Of particular interest are the branched covers $f : S^2 \to S^2$ that are T-equivalent to a polynomial map of one complex variable. These are characterised by the fact that the inverse image of one of the critical values consists of exactly one point (cf. [20]): If one fixes a diffeomorphism $\mathbb{C} \cong S^2 \setminus \infty$ in the range of f, such a branched cover corresponds to a polynomial, unique up to an affine change of the variable.

More generally,

Theorem. *Any branched cover $f : S^2 \to S^2$ is T-equivalent to a rational map.*

Indeed, the complex structure of the range sphere induces via f a complex structure on the domain sphere which is unique up to a linear fractional change of the variable, and with this complex structure on the domain sphere, f is a rational map. □

Thus, a polynomial $f : S^2 \to S^2$ of degree k has passport

$$\Sigma = \{\{d_{11}, \ldots, d_{1l_1}\}, \ldots, \{d_{n1}, \ldots, d_{nl_n}\}, \{k\}\},$$

where

$$\sum_{j=1}^{l_i} d_{ij} = k, \quad i = 1, \ldots, n,$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{l_i}(d_{ij}-1)=k-1$$

(the first condition applies to any passport and reflects the fundamental theorem of algebra; the second condition means that the Euler characteristic of the covering space is equal to 2).

Call a passport satisfying the conditions above a *polynomial passport.*

Conversely, for any polynomial passport Σ there is a branched cover $f: S^2 \to S^2$ with passport Σ (and hence a polynomial with passport Σ). We shall give a proof of this fact in Sec. 3, as the one given in [20] seems to us to be incomplete.

Consider the space $\mathcal{P}$ of polynomials in one complex variable. To each polynomial passport Σ there corresponds a subspace $\mathcal{P}_\Sigma$ of $\mathcal{P}$ consisting of the polynomials with passport Σ. This gives a stratification of $\mathcal{P}$. The path connected strata $\mathcal{P}_\Sigma$ correspond to passports Σ with a unique T-equivalence class. Thus, in this paper we are concerned with conditions on a polynomial passport Σ for $\mathcal{P}_\Sigma$ to be path connected. A much harder problem is the study of the topology of $\mathcal{P}_\Sigma$, say its higher homology groups. These groups seem to be unknown even for the generic stratum in $\mathcal{P}_k$ (the subspace of polynomials of degree $\leq k$), even for small values of k.

In Sec. 4, we give an analytic proof of a result on polynomial maps $P: S^2 \to S^2$ which states, roughly, that if the number of critical values of a polynomial P of degree k is greater than about $3k/4$, then the T-equivalence class of P is determined by the passport of P. In other words, we prove that all strata $\mathcal{P}_\Sigma$ of codimension up to about $k/4$ are path-connected, whereas in Sec. 5 we show that there are strata of codimension about $k/2$ that are not path-connected.

Finally, in Sec. 5, we shall give some examples of polynomial passports Σ corresponding to strata $\mathcal{P}_\Sigma$ that are not path-connected; we shall list some related open problems; and we shall give the T-classification of branched covers of S^2 of degree 3 all of whose critical values have type $\{3\}$, and of branched covers of S^2 of order 4 all of whose critical values have type $\{2,2\}$.

We would like to point out related papers by Arnold (we quote only his first [1] and his last paper [2] on the subject), Barannikov [3], Catanese and Paluszny [7], Gabrielov [11], Glutsyuk [12], Lyashko [16], and Looijenga [14].

2. A Combinatorial Result

We start by giving a combinatorial translation of the statement of Natanzon's theorem [17].

There is a classical one-to-one correspondence between {topological (up to homeomorphism of the domain) classes of k-sheeted covers of S^2 branched over n given points} and {conjugacy classes of homomorphisms of the free group on n given generators to the symmetric group on k elements such that the product $g_1 \cdots g_n$ of the images of the generators is the identity permutation}, with connected covers corresponding to homomorphisms with transitive image; see, for example, [7] or [18] for an exposition suitable for our purposes.

Let $k \in \mathbb{N}$, let $S(k)$ be the permutation group on k elements $\{1, \ldots, k\}$, let $a \in S(k)$ be arbitrary, and let $g_1, \ldots, g_n \in S(k)$ be such that $g_1 \cdots g_n = a$ and such that the group $\Gamma = \langle g_1, \ldots, g_n \rangle < S(k)$ generated by the g_i is transitive. Let $B(n)$ denote the braid group on n strings with the usual generators $\sigma_1, \ldots, \sigma_{n-1}$.

$B(n)$ acts on the set of such n-tuples $(g_1, ..., g_n)$ by

$$\sigma_i(g_i) = g_{i+1}, \ \sigma_i(g_{i+1}) = g_{i+1}^{-1} g_i \, g_{i+1}, \ \text{and} \ \sigma_i(g_j) = g_j \quad \text{for } j \neq i, \ i+1.$$

Call this action ρ.

Note that the product $a = g_1 \cdots g_n$ is invariant under this action of $B(n)$ on $g_1, \ldots, g_n$. Consider two n-tuples of g_i's to be *B-equivalent* if they belong to the same orbit under the action ρ.

In the case when the g_i's are transpositions, we shall represent the data $g_1, \ldots, g_n$ by a graph (the *monodromy graph*) G on k vertices $v_1, \ldots, v_k$ and n edges $e_1, \ldots, e_n$: two vertices v_i and v_j are connected by an edge e_ℓ if and only if g_ℓ transposes i and j. The subscript of the vertex [edge] is called the *label* of the vertex [edge]. Denote by $V(G)$ the set of vertices of G, and by $E(G)$ its set of edges. We shall consider the g_i's as acting on $V(G)$.

Since $\Gamma = \langle g_1, \ldots, g_n \rangle$ is transitive, G is connected. The triple (k, n, a) is called the *type* of G.

The action ρ induces an action of $B(n)$ on the set of such graphs. Say that two graphs are *B-equivalent* if they belong to the same orbit under this action. Clearly, two B-equivalent graphs have the same type.

We shall repeatedly use the following three remarks.

Remark 1. If two edges $e_i, e_{i+1} \in E(G)$ have either no or two vertices in common, then σ_i (and hence σ_i^{-1}) transposes the labels of these edges and leaves the rest of G undisturbed.

Remark 2. If two edges $e_i, e_{i+1} \in E(G)$ have exactly one vertex in common, Fig. 1 shows the action of σ_i and σ_i^{-1} on the portion of G consisting of these two edges; the rest of G is undisturbed.

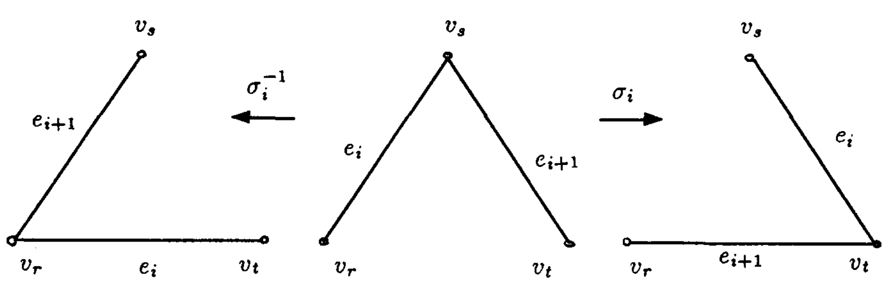


Fig. 1

Given a subset $X \subset E(G)$, let $\bar{X}$ denote the union of X and the subset of $V(G)$ consisting of those vertices that are connected by edges in X.

Remark 3. If $X \subset E(G)$ consists of r edges with labels $t+1, \ldots, t+r$ and if $\bar{X}'$ is obtained from $\bar{X}$ by acting by $\langle \sigma_{t+1}, \ldots, \sigma_{t+r-1} \rangle < B(n)$, then $\bar{X}' \cup (G \backslash \bar{X})$ is B-equivalent to G.

The main result of this section is the following.

Theorem 1. *Given $k \in \mathbb{N}$ and $a \in S(k)$, let G be a graph of type (k, n, a) for some $n \in \mathbb{N}$. Then G is B-equivalent to any other graph of type (k, n, a).*

We start by proving a special case of this theorem. (After writing this paper we discovered that the unpublished part of R. Lawrence's 1989 Oxford University D. Phil. thesis *Homology representations of braid groups* contains a similar Lemma in another context; see Sec. 8.2 in that Thesis.)

Lemma 1. *Let G be a tree on k vertices, i.e. a graph of type $(k, k-1, a)$ for some $a \in S(k)$, and let $v_t \in V(G)$. Then G is B-equivalent to a chain C ending in v_t and such that the labels of the edges are in increasing order: $1, \ldots, k-1$.*

Proof. Assume the result true for trees with $< k-1$ edges, and let G be a tree with $k-1$ edges. Let e_i be a free edge with free vertex v_r. By Remarks 1 and 2, action of σ_i on G shows that G is B-equivalent to a tree with free edge e_{i+1} and free vertex v_r. By consecutively acting with $\sigma_{i+1}, \ldots, \sigma_{k-2}$, we see that G is B-equivalent to a tree G' with free edge e_{k-1} with free vertex v_r. Let v_s be the other vertex of e_{k-1} in G'.

Delete v_r and e_{k-1} from G' and use the inductive hypothesis to show that the remaining tree is B-equivalent (by using only the action of $B(k-2)$) to a chain ending in v_s and whose edges are labelled in increasing order. By Remark 3, G is B-equivalent to a chain C' ending in v_r and whose edges are labelled in increasing order.

If $r = t$, this finishes the proof. If not, it is sufficient to show that C' is B-equivalent to a tree with free vertex v_t and then repeat the above argument to obtain the result.

There are two cases: either v_t is the beginning vertex of C' and we are done; or v_t is the common vertex of two consecutively labelled edges, say e_ℓ and $e_{\ell+1}$, in C'. In the second case, by applying σ_ℓ we see that C' is B-equivalent to a tree with free edge e_ℓ with free vertex v_t. This finishes the proof. □

Remark 4. Lemma 1 implies that

(a) for a graph of type $(k, k-1, a)$, $a \in S(k)$ is a cycle of length k;
(b) the order of the vertices in C is $a^{k-1}v_t, \ldots, av_t, v_t$;
(c) given any vertex v in a tree G, G is B-equivalent to a chain whose edges are labelled in increasing order $1, \ldots, k-1$ and vertices are in the following order: $v, a^{-1}v, \ldots, a^{-k+1}v$. It is easy to adjust the argument of Lemma 1 to show that

if $X \subset E(G)$ for some graph G consists of edges with labels $t+1, \dots, t+k-1$, such that $\bar{X}$ is a tree, and if v is a vertex of $\bar{X}$, then G is B-equivalent to G' obtained from G by replacing $\bar{X}$ by a chain ending [starting] at v with edges labelled in increasing order $t+1, \dots, t+k-1$, whose vertices are consecutively $b^{k-1}v, \dots, v$ $[v, \dots, b^{-k+1}v]$, where $b = g_{t+1} \cdots g_{t+k-1}$, and the rest of G and G' coincide.

Lemma 2. *Let a graph G contain a subset X of r edges labelled $i_1, \dots, i_r$, with $1 \leq i_1 < \cdots < i_r \leq n$. Let $1 \leq j_1 < \cdots < j_r \leq n$. Then G is B-equivalent to a graph G' containing the same set X of edges but with the label j_s replacing the label i_s for $s = 1, \dots, r$.*

Proof. We shall first show that G is B-equivalent to a graph G'' containing the set X of edges but with the label s replacing the label i_s for $s = 1, \dots, r$. Indeed, if $i_1 \neq 1$, by Remarks 1 and 2, G is B-equivalent to a graph in which the label i_1 is replaced by $i_1 - 1$ and all the other edges in X are unchanged. Iterating this process, we can arrange for i_1 to be replaced by 1. Then we can arrange for i_2 to be replaced by 2, ... , then i_r to be replaced by r.

Now we can use the reverse process to first replace the label r by j_r, then $r-1$ by $j_{r-1}, \dots$, and finally 1 by j_1, thus obtaining the required G'. This finishes the proof. □

Lemma 3. *Let a graph G of type (k, n, a) contain at least one cycle Z. Then G is B-equivalent to G' containing a cycle $\{e_1, e_2\}$ of length 2 and with labels 1, 2.*

Proof. By Lemma 2 we can assume that the labels of the edges in Z are $1, \dots, r$ in some order. Assume $r > 2$. Let v_s and v_t be the vertices of e_1. Since $Z \backslash \{e_1\}$ is a tree, it follows by Lemma 1 that it is B-equivalent to a chain C starting at v_s whose edges are labelled in increasing order $2, \dots, r-1$. A subchain C' of C with labels $2, \dots, j$, say, connects v_s to v_t. By Remark 4, C' is equivalent to a chain beginning at v_t in which the edge labelled 2 connects v_t to v_s. Hence G is B-equivalent to a graph with cycle $\{e_1, e_2\}$ of length 2 connecting the vertices v_s and v_t. □

We shall say that G and G' have the *same geography* if G' is obtained from G only by changing the labels of some edges.

Lemma 4. *Let G contain a cycle $\{e_\ell, e_{\ell+1}\}$, $\ell+1 < n$. Then G is B-equivalent to a graph G' with the same geography as G and with cycle $\{e_{\ell+1}, e_{\ell+2}\}$. More precisely, G' differs from G only in that the labels of e_ℓ and $e_{\ell+2}$ have been interchanged.*

Proof. If e_ℓ and $e_{\ell+2}$ have 2 common vertices, there is nothing to prove.

If $e_{\ell+2}$ has no common vertex with $e_{\ell+1}$ (and hence with e_ℓ), then applying Remark 1 twice, first to the pair $e_{\ell+2}$, $e_{\ell+1}$ and then to e_ℓ, $e_{\ell+1}$, we obtain the result.

If $e_{\ell+2}$ has one common vertex with $e_{\ell+1}$ (and hence with e_ℓ), then applying Remark 2 twice, first to the pair $e_{\ell+2}$, $e_{\ell+1}$ and then to e_ℓ, $e_{\ell+1}$, we obtain the result (see Fig. 2). □

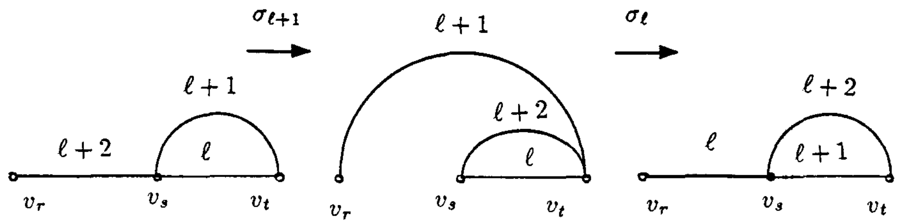


Fig. 2

Remark 5. Note that after this relabelling, the order of the labels in the complement of the cycle $\{e_\ell, e_{\ell+1}\}$ has not changed, i.e. if in G the label of an edge e is less than the label of an edge f, then in G' the label of e is less than the label of f.

Lemma 5. *Let X_1 and X_2 be subsets of $E(G)$ with r and s edges, respectively, such that $\bar{X}_1 \cap \bar{X}_2 = \phi$ and the labels of $X_1 \cup X_2$ are $t+1, \dots, t+r+s$. Then G is B-equivalent to G' with the same geography as G and such that the labels of the edges in X_1 are $t+1, \dots, t+r$, the labels of the edges in X_2 are $t+r+1, \dots, t+r+s$, and the remaining edges have their original labels.*

Proof. Let the labels in X_1 be $j_1, \dots, j_r$ with $j_1 < \dots < j_r$. If $j_1 \neq t+1$, since $\bar{X}_1 \cap \bar{X}_2 = \phi$ and the label $j_1 - 1$ is in X_2, by Remark 1, G is B-equivalent to a graph with the same geography which differs from G in that the labels j_1 and $j_1 - 1$ have been interchanged. Iterating this process, we can reduce j_1 to $t+1$ without changing the geography of G. Then we can reduce j_2 to $t+2, \dots$, and finally j_r to $t+r$. □

Corollary. *Let $X_1, \dots, X_m \subset E(G)$ be such that $\bar{X}_i \cap \bar{X}_j = \phi$ for $i \neq j$, with X_i having r_i edges, and let the labels of the edges in $\cup_{i=1}^m X_i$ run from $t+1$ to $t+\sum_{i=1}^m r_i$. Then G is B-equivalent to a graph with the same geography but in which the labels in X_i run from $t + \sum_{j=1}^{i-1} r_j + 1$ to $t + \sum_{j=1}^{i} r_j$.*

The next lemma allows us to move each vertex of a cycle $Z = \{e_\ell, e_{\ell+1}\}$ in its connected component in $G \backslash Z$.

Lemma 6. *Let G be a graph containing a cycle $Z = \{e_\ell, e_{\ell+1}\}$. Let v_r and v_s be the vertices of e_ℓ (and hence of $e_{\ell+1}$); and let e_i be an edge of G with vertices v_s and v_t. Then G is B-equivalent to a graph G' which differs from G only in that e_ℓ and $e_{\ell+1}$ are replaced by edges e_ℓ and $e_{\ell+1}$ connecting v_r and v_s instead of v_r and v_t.*

Proof. Assume $i > \ell + 1$ (the proof is similar if $i < \ell$). By Lemma 4, G is B-equivalent to a graph G'' with the same geography but in which the labels of Z are $i-2$, $i-1$. Apply σ_{i-1} and σ_{i-2} as in Fig. 3.

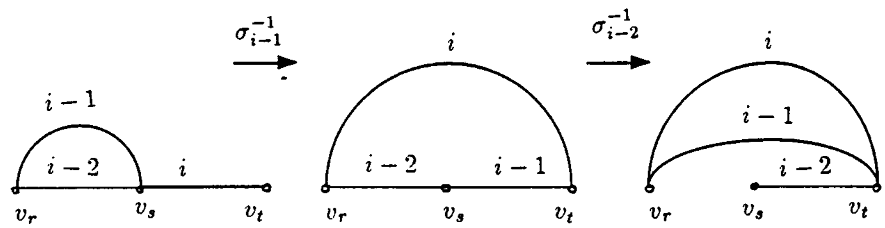


Fig. 3

Then use Lemma 4 to interchange the labels $i-2$ and i, and finally use Lemma 4 again to change the labels $i-1$ and $i-2$ back to $\ell, \ell+1$ without changing the geography. By Remark 5 all the edges in $G \backslash Z$ have their original labels. The resulting graph is the required G'. □

We now define what we mean by a graph G of type (k, n, a) being in standard form. Suppose a is the product of m cycles. A graph G of type (k, n, a) is said to be in *standard form* if:

(i) For each orbit O_i, $1 \le i \le m$, in $V(G)$ of the action of a consisting of r_i vertices, there is a subchain C_i of G consisting of $r_i - 1$ edges connecting the vertices in O_i in the appropriate order; the labels of these edges run from $\sum_{j=1}^{i-1}(r_j - 1) + 1$ to $\sum_{j=1}^{i}(r_j - 1)$.

(ii) The remaining edges in $E(G) \backslash \cup_{i=1}^{m} C_i$ decompose into cycles $\{e_\ell, e_{\ell+1}\}$ of length 2, $\ell = \sum_{i=1}^{m}(r_i - 1) + 1, \sum_{i=1}^{m}(r_i - 1) + 3, \ldots, n-1$.

We are now ready to prove the Theorem. We shall show that G is B-equivalent to a graph of type (k, n, a) in standard form and that any two graphs G' and G'' of type (k, n, a) in standard form are B-equivalent. We shall use induction on the number of edges of G.

So let G be of type (k, n, a). If G is a tree, then by Lemma 1, G is B-equivalent to a chain, which is a standard form.

If G is not a tree, then G contains a cycle. By Lemmas 3 and 4, G is B-equivalent to a graph G' containing a cycle $\{e_{n-1}, e_n\}$. If $G'' = G' \backslash \{e_{n-1}, e_n\}$ is connected, then G'' is a graph of type $(k, n-2, a)$ and by the inductive hypothesis it can be reduced to standard form using the action of $B(n-1)$. Adding $\{e_{n-1}, e_n\}$ to this standard form, we obtain that G can be reduced to standard form.

If G'' is not connected, then it decomposes into two components with edge sets X_1 and X_2 containing s and t edges, respectively. By Lemma 2, we can arrange for the labels in X_1 to be $1, \ldots, s$, and those in X_2 to be $s+1, \ldots, s+t$. By the inductive hypothesis, $\bar{X}_1$ and $\bar{X}_2$ can be reduced to standard form (using the action of $B(s)$ and of $\langle\sigma_{s+1},\ldots,\sigma_{s+t-1}\rangle \cong B(t)$, respectively). By adding $\{e_{n-1}, e_n\}$ and relabelling some of the edges in the cycles of length 2 if necessary, we see that G can be reduced to standard form. It remains to prove that any two graphs G' and G'' of type (k,n,a) in standard form are B-equivalent.

As G'' and G' are both of type (k,n,a), they have the same vertex set, $V(G'') = V(G')$, and for each $i \in \{1,\ldots,m\}$ there is a $j(i) \in \{1,\ldots,m\}$ such that the orbit $O'_{j(i)} = O_i$.

The chains C_i and $C'_{j(i)}$ may start in a different vertex in O_i, but they connect the vertices in O_i in the same cyclic order according to the corresponding cycle of a.

The remaining edges (i.e. those not in the C_i's) are divided into cycles of length 2 and join the various O_i's so that the resulting graph is connected.

By Remark 4, we can have the chains C_i and $C'_{j(i)}$ start at the same vertex (and then they will connect the remaining vertices in O_i in the same order).

By the Corollary to Lemma 5, for each $i \in \{1,\ldots,m\}$, we can relabel the edges in $C'_{j(i)}$ to coincide with the labels in C_i. By Lemma 6, we can move the edges $\{e_{\ell'}, e_{\ell'+1}\}$ to coincide with $\{e_\ell, e_{\ell+1}\}$, without changing the rest of the graph. This finishes the proof of the Theorem. □

3. Polynomial Passports

In this section we shall prove the following theorem (cf. [20], which does not seem to contain a complete proof of this result, though).

Theorem 2. *Let*

$$\Sigma = \{\{d_{11},\ldots,d_{1l_1}\},\ldots,\{d_{n1},\ldots,d_{nl_n}\},\{k\}\},$$

be a polynomial passport, i.e., let

$$(1)\quad \sum_{j=1}^{l_i} d_{ij} = k, \quad i = 1,\ldots,n,$$

and

$$(2)\quad \sum_{i=1}^{n}\sum_{j=1}^{l_i}(d_{ij}-1) = k-1.$$

Then there is a representation σ of the free group $F_n = \langle g_1,\ldots,g_n\rangle$ on n generators into the symmetric group S_k on k elements such that $\sigma(g_i)$ decomposes into cycles of lengths $d_{i1},\ldots,d_{il_i}$, and $\sigma(\prod_{i=1}^{n} g_i)$ is a cycle of length k.

Proof. We shall describe σ as a tree (the *monodromy tree*) T on k vertices and $k-1$ oriented edges, each edge with a label from the set $\{1,\ldots,n\}$: for each d_{ij}, T will have a subchain with $d_{ij}-1$ oriented edges:

$$\underset{v_r}{\bullet} \xrightarrow{i} \underset{v_s}{\bullet} \xrightarrow{i} \cdots \xrightarrow{i} \underset{v_t}{\bullet}$$

meaning that $\sigma(g_i)$ sends v_r to v_s, etc., and v_t to v_r.

Then, by relabelling the edges of T with $\{1, \ldots, k-1\}$ in increasing order along the ordered cycles with increasing labels (so that the product in order of increasing labels of the transpositions in the newly labelled tree is equal to $\sigma(\prod_{i=1}^{n} g_i)$), and by using Remark 4 (a) in Sec. 2, it follows that the product of the generators of F_n gets mapped under σ to a cycle of length k.

We shall build T inductively, starting from the set $V = \{v_1, \ldots, v_k\}$ of vertices, then adding $d_{11} - 1$ oriented edges forming a chain C_{11} as above, each edge with label 1, then adding $d_{12} - 1$ oriented edges forming a chain C_{12} disjoint from C_{11}, each edge with label 1, etc., $d_{1l_1} - 1$ oriented edges in a chain C_{1l_1} disjoint from all the previous ones, each edge with label 1; then we shall add $d_{21} - 1$ oriented edges forming a chain C_{21} each edge with label 2, etc., d_{nl_n} oriented edges in a chain C_{nl_n} disjoint from all the previously built chains C_{nj}, $j = 1, \ldots, l_n - 1$, each edge with label n.

For convenience, we shall assume, for each i, that

$$d_{i1} \geq d_{i2} \geq \cdots \geq d_{il_i},$$

so that those d_{ij}'s that are equal to 1 occur at the end of the sequence $d_{i1}, \ldots, d_{il_i}$.

So let $T_0 = V$. Pick the first d_{11} vertices of T_0 and connect them with oriented edges into a chain as follows:

$$\underset{v_1}{\bullet} \xrightarrow{1} \underset{v_2}{\bullet} \xrightarrow{1} \cdots \xrightarrow{1} \underset{v_{d_{11}}}{\bullet} .$$

Then pick the next d_{12} vertices of V and connect them similarly into a chain, etc. Note that if $d_{1j} = 1$ then we do not add any further edges after the jth step. Because of (1), there are enough new points at each stage of the above construction to build these chains.

We thus obtain a forest $T_1^{l_1}$.

Suppose that the forest T_i^j has been constructed, $j \leq l_i$. We shall show how to construct T_i^{j+1} if $j < l_i$, and T_{i+1}^1 if $j = l_i, i < n$.

Case $j = l_i$. In each path component of T_i^j pick the vertex with smallest subscript. Because of (2), there are at least $d_{i+1\,1}$ path components in T_i^j, and hence at least that many vertices picked. Connect the first $d_{i+1\,1}$ vertices thus obtained in increasing order into an oriented chain, each edge labelled by $i+1$, to obtain the forest T_{i+1}^1.

Case $j < l_i, i < n$. In each path component of T_i^j pick the vertex with smallest subscript that is not adjacent to an edge with label i. We shall show below why there are at least $d_{i\,j+1}$ such vertices. Connect the first $d_{i\,j+1}$ vertices thus obtained in increasing order into an oriented chain, each edge labelled by i, to obtain the forest T_i^{j+1}.

It remains to show that there are at least $d_{i\,j+1}$ vertices as described above. Because of (2), there are at least $d_{i\,j+1}$ path components in T_i^j. If each path component of T_i^j contains a vertex that is not adjacent to an edge with label i, then we are done. If one path component K of T_i^j consists of vertices all adjacent to edges with label i, then it follows from the construction that all the other path components of T_i^j consist of just one vertex, so by (1) there are at least $d_{i\,j+1}$ of them.

Continue this process until all $k-1$ oriented edges are used, thus obtaining the required tree, and hence the required representation of F_n into S_k. □

Note that there is a very simple connection between the graphs in this section and the plane "pictures" from [22]. For a polynomial P of degree k whose critical values are the nth roots of 1, the *picture* of P is the inverse image under P of the unit disc. Given a graph with labelled oriented edges as described in this section, the picture corresponding to the polynomial associated to this graph is topologically the following: take k topological discs (one for each vertex of the graph) on which n points are marked ("the nth roots of 1"). Attach two discs at their points "$\exp 2\pi i s/n$" if the corresponding vertices are joined by an edge with label s; if more than two discs are attached at a point, their corresponding vertices form a chain in the graph; the cyclic order of the discs on the plane is the order of the corresponding vertices on the chain.

4. Analytical Results

The aim of this section is to prove the topological uniqueness of polynomial maps with certain passports.

Theorem 3. *Let $P : S^2 \to S^2$ be a polynomial of degree k, with passport Σ_P. Let $\{b_1^0, \dots, b_m^0\}$ be the set of nonsimple critical values of P, let $a_{i1}^0, \dots, a_{il}^0$ be the critical points of P corresponding to b_i^0, and let $d_{ij}-1$ be the multiplicity of a_{ij}^0. If the defect*

$$d = \sum_{i=1}^{m} \sum_{j=1}^{l_i} d_{ij}$$

of P is $\leq k+1$, then P is T-equivalent to any polynomial with passport Σ_P.

Proof. We need to show that the space of polynomials with passport Σ_P is path connected.

Step 1. Recall the classical proof of the existence and uniqueness of a Lagrange interpolation polynomial L with multiple roots. The problem is to find a polynomial of degree $\leq k$ such that for given x_i, y_{ij}, $0 \leq j \leq r_i$, $1 \leq i \leq s$, $\sum_{i=1}^{s}(r_i - 1) = k-1$, the following are satisfied:

$$(*) \qquad L^{(j)}(x_i) = y_{ij},\ 0 \leq j \leq r_i,\ 1 \leq i \leq s.$$

First of all, the uniqueness of L is clear: if there were two solutions L_1 and L_2 to $(*)$, then their difference $L_1 - L_2$ would be a polynomial of degree $\leq k$ with $k+1$ roots counted with multiplicities.

To prove the existence of $L = \alpha_0 + \alpha_1 x + \cdots + \alpha_k x^k$ satisfying $(*)$, consider $(*)$ as a system of linear equations in the coefficients α_i of L. The matrix of that system is nonsingular because the associated homogeneous system has at most one (and hence exactly one) solution, as noted above. Note that the α_i's are rational functions of the x_i, y_i's.

Step 2. If $d < k+1$, add $k+1-d$ regular values $b^0_{m+1}, \ldots, b^0_{m+k+1-d}$ and pick corresponding points $a^0_{m+1}, \ldots, a^0_{m+k+1-d}$ such that $P(a^0_i) = b^0_i$, $m+1 \le i \le m+k+1-d$.

It follows from *Step 1* that the data

$$b^0_1, \ldots, b^0_{m+k+1-d},\ a^0_{11}, \ldots, a^0_{ml_m},\ a^0_{m+1}, \ldots, a^0_{m+k+1-d}$$

uniquely determine P through the system (we omit the superscript 0 for later convenience):

$$(**)\quad P(a_{ij}) = b_i,\ P^{(j)}(a_{ij}) = 0,\ 1 \le i \le m,\ 1 \le j \le l_j,$$

$$P(a_i) = b_i,\ m+1 \le i \le m+k+1-d.$$

Note that any other choice of $b_i, a_{ij}, a_i \in \mathbb{C}$ with

$$(***)\quad b_i \ne b_{i'},\ a_{ij} \ne a_{i'j'},\ a_i \ne a_{i'} \text{ for } i \ne i',\ j \ne j',$$

also determines a unique polynomial with critical values b_i, $1 \le i \le m$, and corresponding critical points a_{ij} of multiplicity $d_{ij} - 1$.

Let

$$C = \{(a_{11}, \ldots, a_{ml_m}, a_{m+1}, \ldots, a_{m+k+1-d}, b_1, \ldots, b_{m+k+1-d}) = (\mathbf{a}, \mathbf{b});$$

$$a_{ij}, a_i, b_i \in \mathbb{C} \text{ such that } \ (***)\ \text{ holds}\}$$

Denote by $L(\mathbf{a}, \mathbf{b})$ the Lagrange polynomial determined by $(**)$ that corresponds to $(\mathbf{a}, \mathbf{b}) \in C$. The map $L\ : C \to \mathcal{P}_k$ to the space of polynomials of degree $\le k$, parametrised by the coefficients of the polynomials, is rational. The image of L clearly contains all the polynomials with passport Σ_P.

But for some points $(\mathbf{a}, \mathbf{b}) \in C$, $L(\mathbf{a}, \mathbf{b})$ might have passport differing from Σ_p because extra degeneracy might arise (either extra critical points may merge or their corresponding critical values may do so). In $\mathcal{P}_k$, this happens on a union D of affine subspaces of real codimension ≥ 2. Hence, $L^{-1}(D)$ is a subvariety of C, which is proper because $P \notin L^{-1}(C)$.

Thus, $C \setminus L^{-1}(D)$ is path connected, and hence $L(C \setminus L^{-1}(D))$ is also path connected, as required. □

We shall call a polynomial passport satisfying the conditions of Theorem 3 a *Lagrangian passport.*

Note that we actually proved the following stronger

Theorem 4. *For each Lagrangian passport Σ, the corresponding stratum $\mathcal{P}_\Sigma$ of polynomials with passport Σ is biregularly isomorphic to a Zariski open set in affine space.* □

Remark. Note that the passport of a polynomial of degree k with at least $3k/4$ critical values is Lagrangian.

The following theorem was stated in [22].

Theorem 5. *Let $P: S^2 \to S^2$ be a polynomial of degree k with critical values $(b_1^0, \ldots, b_n^0)$ such that to each critical value b_i^0 there corresponds exactly one (possibly multiple) critical point a_i^0. Then P is T-equivalent to any polynomial with passport Σ_P.*

Proof. Let $d_i - 1$ be the multiplicity of a_i^0. Consider the space

$$A = \{\mathbf{a} = (b_0, a_1, \ldots, a_n),\ a_i, b_0 \in \mathbb{C},\ a_i \neq a_j \text{ for } i \neq j,\ 1 \leq i, j \leq n\}.$$

Define a polynomial $Q_{\mathbf{a}}: S^2 \to S^2$ of degree k by

$$Q'_{\mathbf{a}}(x) = \prod_{i=1}^{n}(x - a_i)^{d_i - 1},\ Q_{\mathbf{a}}(0) = b_0.$$

Then $Q: A \to \mathcal{P}_k$ is a map into the space of polynomials of degree $\leq k$ parametrised by their coefficients.

As before, let $D \subset \mathcal{P}_k$ be the codimension ≥ 2 subspace of $\mathcal{P}_k$ of polynomials with higher degeneracies than those of P (due to the fact that the values of some critical points may have merged). Again, $Q^{-1}(D)$ is a proper subvariety of A because $P \notin Q^{-1}(D)$, hence $A \setminus Q^{-1}(D)$ is path connected, as required. □

5. Examples and Questions

We start with the case of polynomials because it is the simplest and also the most interesting one.

Let $\mathcal{P}$ be the space of polynomials of one complex variable, let $\mathcal{P}_k$ be the (complex) $(k+1)$-dimensional linear subspace of $\mathcal{P}$ consisting of polynomials of degree $\leq k$, and for each polynomial passport Σ let $\mathcal{P}_\Sigma$ be the subspace of $\mathcal{P}$ consisting of polynomials with passport Σ.

Theorem 6. *For any polynomial passport Σ, the stratum $\mathcal{P}_\Sigma$ of polynomials with passport Σ is aspherical.*

Proof. Let Σ be a polynomial passport with n_i decompositions D_i of k, $i = 1, \ldots, m$, $D_i \neq D_j$ for $i \neq j$, $\sum_{i=1}^m n_i = n$.

By Theorem 2, at least one representation σ of the free group F_n on n generators into the symmetric group S_k on k elements corresponds to Σ. Two such representations σ will be called equivalent if they differ by an inner automorphism of S_k.

Fix n distinct complex numbers

$$b_{11}, \ldots, b_{1n_1}, \ldots, b_{m1}, \ldots, b_{mn_m}.$$

To each equivalence class of representations for which the first n_1 generators get mapped to permutations of type D_1, the next n_2 generators get mapped to permutations of type D_2, etc., there corresponds a cover $f : S^2 \to S^2$ for which b_{ij} is a critical value of type D_i, which is unique up to a homeomorphism of the domain; i.e., if f_1 and f_2 correspond to the same equivalence class of σ and have the b_{ij}'s as critical values of type D_i, then there is a homeomorphism $g : S^2 \to S^2$ such that $f_1 = f_2 g$. By giving the domain S^2's the complex structure induced by f_1 and f_2, respectively, from the complex structure on the range S^2, f_1 and f_2 are easily seen to be polynomial and g a linear map

$$g = a_1 x + a_2, \quad (a_1, a_2) \in (\mathbb{C} \setminus 0) \times \mathbb{C}.$$

Let F denote the finite set of equivalence classes of such representations σ. Let B be the space of n-tuples

$$(b_{11}, \dots, b_{1n_1}, \dots, b_{m1}, \dots, b_{mn_m})$$

of distinct points in $\mathbb{C}$ factored by the action of $S_{n_1} \times \cdots \times S_{n_m}$ in which S_{n_1} permutes the first n_1 coordinates, S_{n_2} permutes the next n_2 coordinates, etc. It is well known that B is a $K(B(n_1, \dots, n_m), 1)$, where $B(n_1, \dots, n_m)$ is the group of braids on $n = n_1 + \cdots + n_m$ strings, the first n_1 colored in color 1, the next n_2 colored in color 2, etc.

Let

$$v : \mathcal{P}_\Sigma \to B$$

be the map that sends a polynomial P with passport Σ to its n-tuple $(b_{11}, \dots, b_{mn_m})$ of critical values, where the n_1 critical values of type D_1 provide the first n_1 coordinates, the n_2 critical values of type D_2 the next n_2 coordinates, etc.

Then v is a fibre bundle, and by the above, the fibre of v is $F \times (\mathbb{C} \setminus 0) \times \mathbb{C}$. As the fibre and base are aspherical, so is the total space $\mathcal{P}_\Sigma$, which was to be proved. □

In the case in which $\mathcal{P}_\Sigma$ is path connected, Theorem 6 implies that $\mathcal{P}_\Sigma$ is a $K(\pi_\Sigma, 1)$ for some group π_Σ which is an extension

$$0 \to \mathbb{Z} \to \pi_\Sigma \to \pi \to 0,$$

where π is a subgroup of index $|F|$ of the braid group $B(n_1, \dots, n_m)$.

Problem. *Compute the homology of* π_Σ.

If P has at least about $3k/4$ critical values, then P is Lagrangian, and hence the stratum containing P is path connected.

Thus any stratum of complex codimension up to about $k/4$ in $\mathcal{P}_k$ is path connected.

At the other end of the interval of codimensions, if P has just one critical value, then the corresponding stratum is path connected (say, by Theorem 5).

But if P has 2 critical values, the corresponding stratum consists, in general, of many path components. Such polynomials are being extensively studied (see the excellent survey article by Shabat and Zvonkin [19]) because of the following remarkable theorem of Belyi [4]:

Belyi's Theorem. *For a Riemann surface X, a meromorphic function $f : X \to S^2$ with only three critical values 0, 1 and ∞ exists if and only if the surface X is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers.*

For a polynomial with just two critical values, say $+1, -1$, the inverse image of a segment connecting $+1$ and -1 is a plane tree ("dessin d'enfant" in Grothendieck's terminology [**Gr**]) with each vertex colored white or black depending on whether it maps to $+1$ or -1. These trees are closely related to the "pictures" considered in [22] in the case of two critical values.

Define c_k to be the least integer such that for polynomials of degree k with $\geq c_k$ critical values the corresponding stratum in $\mathcal{P}$ is path connected.

Question. *What is the value of c_k?*

As noted above, c_k is less than about $3k/4$.

The following example shows that c_k is greater than about $k/2$.

Example 1. Consider the polynomial passport (from now on, we shall omit the 1's in the notation for passports)

$$\Sigma = \{\{2\}, \{2,2\}, \ldots, \{2,2\}, \{k\}\}$$

($m = (k-2)/2$ occurences of $\{2,2\}$). In the notation of Sec. 3, the polynomials P and Q with passport Σ that correspond to the representation of the free group F on $m+1$ generators into the symmetric group S_k on k elements given by the trees

$$\bullet \xrightarrow{1} \bullet \xrightarrow{2} \cdots \xrightarrow{m} \bullet \xrightarrow{m+1} \bullet \xrightarrow{m} \bullet \xrightarrow{m-1} \cdots \xrightarrow{1} \bullet$$

and

$$\bullet \xrightarrow{1} \bullet \xrightarrow{2} \cdots \xrightarrow{m} \bullet \xrightarrow{1} \bullet \xrightarrow{2} \cdots \xrightarrow{m} \bullet \xrightarrow{m+1} \bullet$$

(we assume $m \geq 2$) are not topologically equivalent. Indeed, one can show that P can be represented as $P_1((x+b)^2)$ for some polynomial P_1 of degree $m+1$ with m different critical values, whereas Q cannot be so represented. Combinatorially, the tree corresponding to P admits a map to a tree with m edges under which edges with the same label are identified, whereas the tree corresponding to Q does not admit such a map. As this property is invariant under the action of the braid group, the result follows.

We conjecture that a combinatorial proof of Theorem 3 might lead to an improved upper bound for c_k of about $2k/3$.

Problem. *Find a combinatorial proof of Theorem 3.*

Example 2. In Example 1 we saw that one cannot say by looking at a polynomial passport Σ whether Σ is the passport of a polynomial of the form $P(x^2+a)$. It is natural to ask whether one can characterise the passports of polynomials of degree k of the form $(P(x))^2+b$, or, more generally, of the form

$$(*) \quad Q = P_1^{m_1}\cdots P_l^{m_l} + b, \quad m_i \geq 2, i = 1,\dots,l.$$

The answer is "yes". Namely, a necessary and sufficient condition for a polynomial passport Σ to be the passport of a polynomial of the form $(*)$ is the following: $(**)$ there is an integer $r \geq 1$ and integers $\alpha_{ij} \geq 0$ for $i = 1,\dots l$, $j = 1,\dots r$, such that one of the partitions of k in Σ, call it $\mathcal{S}$, has the form

$$\mathcal{S} = \{\alpha_{11}m_1 + \cdots + \alpha_{1l}m_l, \dots, \alpha_{r1}m_1 + \cdots + \alpha_{rl}m_l\}.$$

Indeed, if Q has the form $(*)$, then each root a of P_j of multiplicity $\alpha_j(a)$ is a critical point of Q of order

$$\alpha_1(a)m_1 + \cdots + \alpha_l(a)m_l,$$

and the corresponding critical value is b. So the partition of k corresponding to the critical value b is as in $(**)$.

Conversely, if Q has passport of the form $(**)$, then there exist polynomials $P_1,\dots,P_l$ such that Q has the form $(*)$; indeed, let b be the critical value of Q corresponding to the partition $\mathcal{S}$. Let $a_1,\dots,a_r$ be the critical points of Q corresponding to b. By assumption, a_i is a critical point of Q of order $\sum_{j=1}^{l}\alpha_{ij}m_j$. Let $P_1,\dots,P_l$ be such that a_i is a zero of multiplicity α_{ij} of P_j. These are the required polynomials.

Remark that if all the other critical values of Q are simple, then Σ is a Lagrangian passport, hence there is a unique class of polynomials corresponding to it.

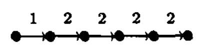
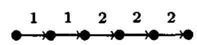
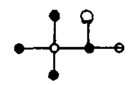
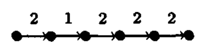
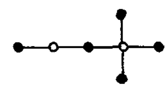
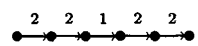
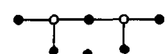
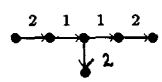
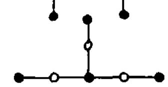
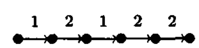
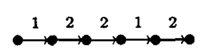
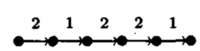
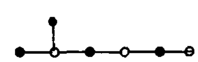
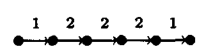
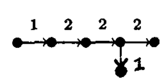
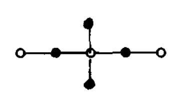
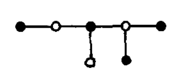
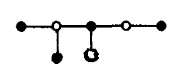
As noted in [22], for polynomials P of degree ≤ 5, the corresponding stratum in $\mathcal{P}$ is path connected. For polynomials of degree 6, the list of T-equivalence classes given in [22] is incomplete, so we give the full list here.

Example 3. T-equivalence classes of polynomials of degree 6.

Case of 1 critical value. Clearly, there is only one class.

Case of 2 critical values. This is the very interesting case discussed in [19]. Table 1 lists all classes of such polynomials by giving their passports, their monodromy trees as described in Sec. 3, their "dessins d'enfant" (see also [19]), and whether or not they satisfy the conditions of Theorems 3 and/or 5.

Table 1.

Passport	Number of classes	Monodromy trees	Dessins d'enfant	Satisfies conditions of Thm(s)
{{2}, {5}}	1			3, 5
{{3}, {4}}	1			3, 5
{{2}, {2, 4}}	1			3
{{2}, {3, 3}}	1			3
{{3}, {2, 2, 2}}	1			
{{2, 2}, {2, 2, 2}}	1			
{{2, 2}, {3, 2}}	3			
				
				
{{2, 2}, {4}}	2			
				
{{3}, {3, 2}}	2			
				

Case of 3 critical values. These polynomials have one of the following passports:

$$\{\{2\},\{2\},\{4\},\{6\}\},\{\{2\},\{2\},\{3,2\},\{6\}\},\{\{2\},\{2\},\{2,2,2\},\{6\}\},$$

$$\{\{2\},\{3\},\{3\},\{6\}\},\{\{2\},\{3\},\{2,2\},\{6\}\},\{\{2\},\{2,2\},\{2,2\},\{6\}\}.$$

All but the last are Lagrangian passports, so to each of them there corresponds a single T-equivalence class of polynomials.

The last is a special case ($m = 2$) of Example 1, and therefore there are at least two classes corresponding to this passport. Action by the braid group $B(3)$ shows that there are exactly two.

Case of 4 critical values. There are only two passports, $\{\{3\},\{2\},\{2\},\{2\},\{6\}\}$ and $\{\{2,2\},\{2\},\{2\},\{2\},\{6\}\}$, and they are both Lagrangian, so for each of these there is only one class.

Case of 5 critical values. This is the classical case studied by Clebsch who proved that there is only one class.

Going back to meromorphic functions $f : M \to S^2$, here are two more examples.

Example 4. Here is the T-classification of 3-sheeted branched covers of S^2 with n critical values, each of type $\{3\}$.

Given such a cover f, under the corresponding representation σ of the free group $F_n = \langle g_1, \dots, g_n\rangle$ on n generators into the symmetric group S_3 on 3 elements mentioned at the beginning of Sec. 2, let n_1 of the g_i's get mapped to the permutation $a = (1,2,3)$, and let n_2 of the g_i's get mapped to $a^2 = (1,3,2)$. As $\sigma(\prod_{i=1}^n g_i)$ is the identity permutation,

$$n_1 - n_2 \equiv 0 \mod 3.$$

As $\sigma(F_n)$ is commutative, the action of the braid group on it is trivial. Also, an inner automorphism of S_3 (i.e. a renumbering of the sheets of the cover), does not change the unordered pair $\{a, a^2\}$. Thus, two such covers are T-equivalent if and only if they are equivalent as covers: they must have the same number n of critical values, and the same unordered pair $\{n_1, n_2\}$.

Hence, for each n, the number t of T-equivalence classes of such covers is equal to the number of ways of representing n as $n_1 + n_2$ with $n_1 - n_2 \equiv 0 \mod 3$.

For $n < 2$, $t = 0$. One can show that for $n \geq 2$, if n is represented as $n = 6q + c$, $c = 0,1,2,3,4,5$, then $t = q + 1$ if $c \neq 1$ and $t = q$ if $c = 1$.

Example 5. Here is the T-classification of 4-sheeted branched covers of S^2 with n critical values, each of type $\{2,2\}$.

Given such a cover f, under the corresponding representation σ of the free group F_n on n generators $g_i, i = 1, \dots, n$, into the symmetric group on 4 elements mentioned at the beginning of Sec. 2, let n_1 of the g_i's get mapped to $a = (12)(34)$, n_2 to $b = (13)(24)$, and n_3 to $c = (14)(23)$.

The subgroup $\langle a, b, c\rangle$ of S_4 is commutative, hence under the action of the braid group $B(n)$ on these covers the numbers n_1, n_2, n_3 do not change. Also, the inner automorphisms of S_4 leave the set $\{a, b, c\}$ invariant and moreover any permutation of this set of three elements is induced by an inner automorphism of S_4.

As $\sigma(\prod_{i=1}^{n} g_i)$ is the identity permutation on 4 elements, either n_1, n_2, n_3 are all even or they are all odd. Moreover, at least two of them must be positive for the covering space to be connected.

So the number t of T-equivalence classes of such branched covers is equal to the number of ways of representing n as a sum of three natural numbers of the same parity, two of which are positive.

Using standard combinatorial calculations (cf., e.g., Example 15.1 , pp. 132-133, of the book by van Lint and Wilson [21]), one obtains the following values of t as a function of n.

1. If n is even, then $t(n)$ is the least integer greater than

$$\frac{n(n+12)}{48} - 1.$$

2. If n is odd, then $t(n)$ is the least integer greater than

$$\frac{(n-3)(n+9)}{48}.$$

Remark that for n odd, all covers of S^2 with such a passport are connected, while for n even, there is also one additional nonconnected cover with the same passport; hence the -1 term in the formula for $t(n)$ for n even.

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Note Added in Proof

After this paper was accepted for publication, the authors found out that another proof of Theorem 2 is contained in a paper by A. L. Edmonds, R. S. Kulkarni, and R. E. Stong [Realizability of branched covers of surfaces, Trans. A.M.S. Vol. 282 (1984), 773–790]. We thank Allan Edmonds for bringing this paper to our attention.

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