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GENERALIZED ROLLE THEOREM IN $\mathbb{R}^n$ AND $\mathbb{C}$

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ABSTRACT. The Rolle theorem for functions of one real variable asserts that the number of zeros of f on a real connected interval can be at most that of f' plus 1. The following inequality is a multidimensional generalization of the Rolle theorem: if $\ell\,[0,1] \to \mathbb{R}^n$, $t \mapsto \mathbf{x}(t)$, is a closed smooth spatial curve and $\mathbf{L}(\ell)$ is the length of its spherical projection on a unit sphere, then for the *derived* curve $\ell'\,[0,1] \to \mathbb{R}^n$, $t \mapsto \dot{\mathbf{x}}(t)$, the following inequality holds: $\mathbf{L}(\ell) \leqslant \mathbf{L}(\ell')$. For the analytic function $F(z)$ defined in a neighborhood of a closed plane curve $\Gamma \subset \mathbb{C} \simeq \mathbb{R}^2$ this inequality implies that $\widetilde{\mathbf{V}}_\Gamma(F) \leqslant \widetilde{\mathbf{V}}_\Gamma(F') + \varkappa(\Gamma)$, where $\widetilde{\mathbf{V}}_\Gamma(F)$ is the total variation of the argument of F along Γ, and $\varkappa(\Gamma)$ is the integral absolute curvature of Γ.

As an application of this inequality, we find an upper bound for the number of complex isolated zeros of quasipolynomials. We also establish a two-sided inequality between the variation index $\widetilde{\mathbf{V}}_\Gamma(F)$ and another quantity, called the *Bernstein index*, which is expressed in terms of the modulus growth of an analytic function.

1. INTRODUCTION

The classical Rolle theorem asserts that the derivative of the smooth real function f must vanish somewhere between any two zeros of this function. As a consequence, the number $N_K(f)$ of real zeros of f on the segment $K \subset \mathbb{R}$ does not exceed $N_K(f') + 1$, and this inequality remains valid if we count zeros with their multiplicities. The simplest examples show that the straightforward generalization of this principle for complex-valued functions fails. Thus one has to look for a proper substitute for $N_K(\cdot)$ that would be defined for functions analytic in some open domain $U \subset \mathbb{C}$. To be suitable

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for applications, such a substitute (denote it for instance by $\mathcal{R}_U(f)$) has to possess the following list of properties.

1.1. A priori requirements for a generalized Rolle theory.

(1) If f is an analytic function in U, then the number of zeros of f in U (or eventually, in the smaller domain $U' \subset U$) admits an upper estimate in terms of $\mathcal{R}_U(f)$ and some geometric characteristics of U (and U', if necessary);
(2) If f, g are two analytic functions, then $\mathcal{R}_U(f \cdot g)$ admits a two-sided estimate in terms of $\mathcal{R}_U(f)$ and $\mathcal{R}_U(g)$ and, eventually, the geometry of U;
(3) $\mathcal{R}_U(f)$ admits an upper estimate in terms of $\mathcal{R}_U(f')$ and, if necessary, the geometry of U.

If all these requirements are met, then one can build a reasonable theory resulting in upper bounds for the number of zeros of analytic functions satisfying certain types of differential equations.

1.2. The Voorhoeve index and the Bernstein index as generalized Rolle theories. There were several attempts to construct a quantity satisfying the above conditions. In [7] Marc Voorhoeve observes that if f is a function analytic in a small neighborhood of a real interval $[0,1]$ and has no zeros on this interval then the total variation of argument of this function, scaled for convenience by a factor of 2π,

$$\mathbf{V}_{[0,1]}(f) = \frac{1}{2\pi}\int_0^1 \left|\frac{d}{dt}\operatorname{Arg} f(t)\right| dt = \frac{1}{2\pi}\int_0^1 \left|\operatorname{Im}(f'/f)\right| dt \tag{1.1}$$

admits an extension by continuity for the class of meromorphic functions eventually having zeros on $[0,1]$, and

$$\mathbf{V}_{[0,1]}(f) \leqslant \mathbf{V}_{[0,1]}(f') + \frac{1}{2}. \tag{1.2}$$

Being subadditive in the sense of (1.8) below, this index clearly satisfies the second requirement from the above list. Now for the polygonal domain $Q \subset \mathbb{C}$ bounded by the closed polyline $\Gamma = \partial Q$ one can define $\mathbf{V}_Q(f)$ as the total variation of the argument of f along the boundary Γ, and by the argument principle $\mathbf{V}_\Gamma(f)$ majorizes the number of zeros of f in Q. Inequality (1.1) implies that for an n-sided closed polyline Γ we have

$$\mathbf{V}_\Gamma(f) \leqslant \mathbf{V}_\Gamma(f') + \frac{n}{2}. \tag{1.2_n}$$

Thus one has a generalized Rolle theory, and it was successfully used in [7], [8] for finding upper bounds for the number of complex zeros of quasipolynomials and some other similar classes of functions (see below) defined by linear ordinary differential equations with constant or rational coefficients. At that time these were the best known estimates.

However, the Voorhoeve index $\mathbf{V}(\cdot)$ is not convenient if one works with functions defined by equations with general variable coefficients. In [6] another quantity, called the *Bernstein index*, was introduced. To define the Bernstein index, one needs, in fact, not one but two nested sets, and if U is an open simply connected domain and $K \Subset U$ is its proper subset at a positive distance from the boundary ∂U and with nonempty interior, then for any function f, analytic on the closure $\overline{U} = U \bigcap \partial U$, the quantity

$$\mathbf{B}_{K,U}(f) = \ln \max_{z \in \overline{U}} |f(z)| - \ln \max_{z \in K} |f(z)| \tag{1.3}$$

is a nonnegative number by the maximum modulus principle. A simple generalization of the Jensen inequality shows that the number of zeros of f in K can be at most $\gamma \mathbf{B}_{K,U}(f)$, where $\gamma = \gamma(K, U) < \infty$ is a constant easily described in terms of the relative position of K inside U. This definition, though formally dependent on the pair of sets $K \Subset U$, is essentially independent of K; if one replaces K by another subset $K' \Subset U$ with a nonempty interior, then the Bernstein index with respect to this new pair will be equivalent to the old one as a functional on the space of functions analytic on $\overline{U}$ (see Sec. 4 for the precise formulation). Finally, one can make sure that the other two preconditions for the Rolle theory are also satisfied.

The main result of this paper is twofold. First we establish by geometrical means the inequality generalizing (1.2) for arbitrary curves in $\mathbb{R}^n$ and apply this generalized Rolle inequality to get better upper bounds for the number of isolated zeros of complex quasipolynomials. Second, we show that the Bernstein index (1.3) is, in some sense, equivalent to the properly generalized Voorhoeve index. The precise formulations follow.

1.3. The total variation of the argument and its properties.

If $\Gamma \subset \mathbb{C}$ is a parametrized plane curve not passing through the point $a \in \mathbb{C}$, then its rotation about that point is defined as $\operatorname{Arg}(z(1)-a)-\operatorname{Arg}(z(0)-a) = \int_0^1 \frac{d}{dt} \operatorname{Arg}\big(z(t)-a\big)\,dt$ for any choice of a continuous branch of $\operatorname{Arg}(z-a)$ (and any parametrization $[0,1] \ni t \mapsto z(t) \in \mathbb{C}$ of Γ). We define the *total absolute rotation* of Γ around a as the total variation of the function $\operatorname{Arg}\big(z(t)-a\big)$, i.e., the integral $\int_0^1 \left|\frac{d}{dt} \operatorname{Arg}\big(z(t)-a\big)\right| dt$. For the (piecewise C^2)-smooth curve $\Gamma \subset \mathbb{C}$ let $\varkappa(\Gamma)$ be the total absolute curvature of Γ defined as the absolute rotation of its tangent vector about the origin: if $\varkappa(z)$ is the curvature of Γ

at the point $z \in \Gamma$, then one can easily verify that

$$\varkappa(\Gamma) = \int_\Gamma |\varkappa(z)|\,|dz| = \text{(total absolute rotation of the tangent vector)}.$$

For the function $f\ \Gamma \to \mathbb{C}$ that has no zeros on Γ we define the index $\mathbf{V}_\Gamma(f)$ in the same way as in (1.1), by the equivalent relations

$$\begin{aligned}\mathbf{V}_\Gamma(f) &= \frac{1}{2\pi} \times \text{(total variation of } \operatorname{Arg} f \text{ along } \Gamma) = \\ &= \frac{1}{2\pi} \times \text{(absolute rotation of the image } f(\Gamma) \text{ about zero)}.\end{aligned}$$

This definition extends by continuity to functions meromorphic in an open neighborhood of Γ in the same way as in [7]. Note that if $t \mapsto z(t)$ is the parametrization of the curve Γ, then

$$\varkappa(\Gamma) = 2\pi \mathbf{V}_\Gamma(\dot z). \tag{1.4}$$

Let $f\,(\mathbb{C}, \Gamma) \to \mathbb{C}$ be analytic in a neighborhood of Γ without zeros on Γ and $\theta_t \in [0, \pi]$ be the (geometric, i.e., nonoriented) angle between $f'\big(z(t) =\big)\cdot \dot z(t)$ and $f\big(z(t)\big)$:

$$\theta_t = \operatorname{Arg}_+\left(\frac{f'\big(z(t)\big)\cdot \dot z(t)}{f\big(z(t)\big)}\right), \qquad \operatorname{Arg}_+(w) := \arccos\big(\operatorname{Re} w/|w|\big),$$

where $\operatorname{Arg}_+(w)$ is the absolute value of the angle between $w \in \mathbb{C}$ and the positive semiaxis, and $f'(z) = \frac{d}{dz} f(z)$. Obviously, θ_0, θ_1 do not depend on the parametrization of Γ provided that the orientation remains the same, and $\theta_0 - \theta_1$ is preserved even by orientation-reverting parametrizations, being thus an invariant of a function analytic around a nonparametrized curve.

The following result is a generalization of the Rolle theorem for complex-valued analytic functions.

Theorem 1 (Rolle theorem for analytic functions). *If the function $f(\mathbb{C}, \Gamma) \to \mathbb{C}$ is meromorphic in some open neighborhood of the curve $\Gamma \subset \mathbb{C}$, then*

$$\mathbf{V}_\Gamma(f) \leqslant \mathbf{V}_\Gamma(f') + \frac{1}{2\pi}\varkappa(\Gamma) + \frac{1}{2\pi}(\theta_0 - \theta_1). \tag{1.5}$$

Corollary 1. *If Γ is a closed smooth curve, then*

$$\mathbf{V}_\Gamma(f) \leqslant \mathbf{V}_\Gamma(f') + \frac{\varkappa(\Gamma)}{2\pi}. \tag{1.6}$$

Indeed, in this case $\theta_1 = \theta_0$.

Corollary 2. *If Γ is the boundary of a convex bounded plane domain, then*

$$\mathbf{V}_\Gamma(f) \leqslant \mathbf{V}_\Gamma(f') + 1. \tag{1.7}$$

Note that this estimate is always better than that of (1.2_n), since even for a triangle (the smallest possible $n = 3$) one has in (1.7) the term 1 instead of $\frac{3}{2}$. Obviously, inequality (1.7) is sharp.

Since the argument of a product is the sum of the arguments, $\operatorname{Arg}(fg) = \operatorname{Arg} f + \operatorname{Arg} g$ (assuming a proper choice of continuous branches), the triangle inequality immediately implies that

$$|\mathbf{V}_\Gamma(fg) - \mathbf{V}_\Gamma(f)| \leqslant \mathbf{V}_\Gamma(g), \tag{1.8}$$

$$\mathbf{V}_\Gamma(f^{-1}) = \mathbf{V}_\Gamma(f), \tag{1.9}$$

(cf. [7], and by the argument principle the number of isolated zeros of a function *analytic* in U does not exceed $\mathbf{V}_\Gamma(f)$. Thus the correspondence $f \mapsto \mathbf{V}_\Gamma(f)$ is indeed a generalized Rolle theory in the sense described above.

We prove Theorem 1 in Sec. 2 as a two-dimensional special case of a more general geometric inequality (Theorem 4) valid for curves in $\mathbb{R}^n$.

1.4. Application to zeros of quasipolynomials. As an application of Theorem 1, we obtain an upper bound for the number of complex isolated zeros of quasipolynomials, which is asymptotically better than that from [7].

Definition. Let Λ be a finite subset of $\mathbb{C}$. A *quasipolynomial* with the spectrum Λ is a finite sum of the form

$$f(z) = \sum_{\mu \in \Lambda} p_\mu(z) \exp \mu z, \qquad p_\mu \in \mathbb{C}[z].$$

The *degree* of a quasipolynomial is the sum $\sum_\mu (1 + \deg p_\mu)$.

For the fixed point set Λ all quasipolynomials of degree d with the spectrum Λ constitute a d-dimensional (complex) linear space. Recall that a linear d-dimensional space of functions is called a *Chebyshev family* in the domain U if any function from this family has no more than $d - 1$ isolated roots in U (this definition makes sense both in real and complex settings). We say that $s \in \mathbb{N}$ is the *Chebyshev excess* of a d-dimensional family if the number of roots of any function from that family can be at most $d-1+s$. By $s(d, U, \Lambda)$ we denote the Chebyshev excess for the linear space of quasipolynomials of degree d with the spectrum Λ in the domain U.

For any bounded subset $U \subset \mathbb{C}$ we denote by $\operatorname{diam}_{\mathbb{C}}(U) = \sup_{z,w \in U} |z - w|$ the Euclidean diameter of U. Any bounded convex set $U \subset \mathbb{C}$ defines a *seminorm* in $\mathbb{C} \simeq \mathbb{R}^2$ as

$$\|w\|_U = H_U(w) + H_U(-w), \qquad \text{where} \quad H_U(w) = \sup_{z \in U} \operatorname{Im}(zw).$$

This seminorm is invariant by translation, $\|\cdot\|_U = \|\cdot\|_{a+U}$ for any $a \in \mathbb{C}$, and $\|w\|_U \leqslant |w| \operatorname{diam}_{\mathbb{C}} U$. If $U \subseteq \mathbb{R}$ and $w \in \mathbb{R}$, then $\|w\|_U = 0$.

For the finite set Λ we denote by $\#\Lambda$ the number of distinct points in Λ and by $\mathcal{L}(\Lambda)$ the length of the shortest plane polyline passing through all points of Λ and let $\mathcal{L}_U(\Lambda)$ be the length of the shortest polyline with respect to the seminorm $\|\cdot\|_U$. Obviously, $\mathcal{L}(\Lambda) \leqslant (\#\Lambda - 1) \cdot \operatorname{diam}_{\mathbb{C}}(\Lambda)$, but, in fact, a stronger estimate can be easily proved; see Sec. 3. Our main result concerning complex zeros of quasipolynomials is the following inequality proved and discussed in Sec. 3.

Theorem 2. *If U is a bounded convex domain, then*

$$\begin{aligned} s(d,\Lambda,U) &\leqslant \frac{1}{\pi}\cdot \mathcal{L}_U(\Lambda) &\quad (1.10a)\\ &\leqslant \frac{1}{\pi}\mathcal{L}(\Lambda)\cdot \operatorname{diam}_{\mathbb{C}}(U) &\quad (1.10b)\\ &\leqslant \frac{2}{\pi}\big(\sqrt{\#\Lambda}+1\big)\cdot \operatorname{diam}_{\mathbb{C}}(\Lambda)\cdot \operatorname{diam}_{\mathbb{C}}(U). &\quad (1.10c)\end{aligned}$$

1.5. Comparison between two theories. Finally, we compare two different generalized Rolle theories and prove that they are essentially equivalent.

Theorem 3. *If $K \Subset U$ is a pair of subsets as in the definition of the Bernstein index and $\Gamma \Subset U$ is a closed curve at a positive distance from the boundary of U, then, for any f analytic on $\overline{U}$, we have*

$$\mathbf{V}_\Gamma(f) \leqslant \alpha\, \mathbf{B}_{K,U}(f), \tag{1.11}$$

where $\alpha = \alpha(K,U,\Gamma) < \infty$ is a constant completely determined by the geometry of the sets.

Conversely, if $\Gamma = \partial U'$ is a simple closed curve containing a simply connected domain U strictly inside, $U \Subset U'$, then for any $K \Subset U$ and for any f analytic on the closure $\overline{U'}$, we have

$$\mathbf{B}_{K,U}(f) \leqslant \beta \mathbf{V}_\Gamma(f), \tag{1.12}$$

where $\beta = \beta(K,U,\Gamma) < \infty$ is yet another geometric constant.

1.6. Remarks on geometric constants. In the above exposition we have already introduced some *geometric constants* such as the total absolute rotation of a curve, the Euclidean diameter of a set, etc., and several more will appear in subsequent sections. We want to emphasize that all these geometric constants are explicit quantities which can be computed by elementary methods for any reasonable domain (or pair of domains).

2. Rolle theorems in $\mathbb{R}^n$ and $\mathbb{C}$ via integral geometry and the game of pursuit

2.1. The inequality of the lengths of spherical projections: formulation of the result. Consider in $\mathbb{R}^n$ a parametrized nonsingular C^2-smooth curve, $\ell\ t \mapsto \mathbf{x}(t)$ defined on the interval $t \in [0,1]$ and not passing through the origin. The *velocity function* $t \mapsto \dot{\mathbf{x}}(t)$ can be considered as a C^1-smooth *derived curve* ℓ', and since ℓ is smooth and nonsingular, ℓ' does not pass through the origin either. We denote by $\mathbf{L}(\ell)$ the length of the spherical projection of ℓ, i.e., the curve $[0,1] \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$, $t \mapsto \mathbf{x}(t)/\|\mathbf{x}(t)\|$ (called also the *spherical indicatrix* of the curve ℓ). The length $\mathbf{L}(\ell')$ of the spherical projection of ℓ' is defined in the same way. Note that the (standard) sphere is a Riemannian manifold (the metric is inherited from $\mathbb{R}^n$). The (spherical) distance will be denoted by $\operatorname{dist}_{\mathbb{S}^{n-1}}(\cdot,\cdot)$ and we will use the same symbol for the angular distance between any two points in $\mathbb{R}^n \smallsetminus \{0\}$. By $d\sigma_{n-1}$ we denote the associated area form on $\mathbb{S}^{n-1}$, so that $\sigma_{n-1} = \int_{\mathbb{S}^{n-1}} d\sigma_{n-1}$ is the area of the unit sphere.

Theorem 4 (Rolle theorem in $\mathbb{R}^n$). *The spherical lengths of projections of the smooth curve $\ell\ t \mapsto \mathbf{x}(t)$ and its derived curve $\ell'\ t \mapsto \dot{\mathbf{x}}(t)$ are related by the inequality*

$$\mathbf{L}(\ell) \leqslant \mathbf{L}(\ell') + \Phi(0) - \Phi(1), \tag{2.1}$$

where $\Phi(t) = \operatorname{dist}_{\mathbb{S}^{n-1}}(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ is the spherical (angular) distance between the projections of $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$.

Corollary 1. *If the curve ℓ is closed, i.e., $\mathbf{x}(0) = \mathbf{x}(1)$, then $\mathbf{L}(\ell) \leqslant \mathbf{L}(\ell')$.*

Corollary 2. *For any curve ℓ one has $\mathbf{L}(\ell) \leqslant \mathbf{L}(\ell') + \pi$.*

Indeed, the spherical distance between any two points does not exceed π.

The quantity $\mathbf{L}(\ell)$ has the natural geometric meaning of *integral rotation* of the curve ℓ about the origin: for $n = 2$ the value $\mathbf{L}(\ell)$ is the total variation of $\operatorname{Arg}(\mathbf{x}(t))$ on the interval $[0,1]$, see 2.5.

On the other hand, from the first Frenet formula it follows that the velocity vector of the derived curve ℓ' has the length $|\varkappa(t)|$, where $\varkappa(t)$ is the Euclidean curvature of the original curve ℓ at the point $\mathbf{x}(t)$. Thus $\mathbf{L}(\ell')$ is the (absolute) integral curvature of ℓ, and the assertion of Corollary 1 can be reformulated as follows.

Corollary 3. *The Rotation of a space closed curve about any point does not exceed the integral curvature of this curve.*

The assertion of Corollary 2 admits a similar reformulation for nonclosed curves.

We give two independent proofs of Theorem 4. The first one, given in 2.2, shows intrinsic connections between this inequality and the classical Rolle theorem in one variable (similar ideas were used by J. Milnor in [5]). The second proof from 2.3 is, in fact, more general and extends to the case of curves on an arbitrary Riemannian manifold. As a corollary, we deduce Theorem 1 in 2.5.

Remark. There are other Rolle-type theorems for curves in $\mathbb{R}^n$, of completely different nature (they deal with intersections of Pfaffian varieties; inequality (2.3) can be considered as a trivial case of this theorem). This other direction is exposed in [3].

2.2. Integral geometry. We start with the Rolle theorem in one variable, but add a certain correction term. Let $f\ [0,1] \to \mathbb{R}$ be a smooth function in one variable and let $N(f)$ stand for the number of isolated zeros of f on $[0,1]$, counted with their multiplicities.

Lemma 1 (A refined Rolle theorem in one variable).

$$N(f) \leqslant N(f')+\varphi(0)-\varphi(1), \quad \textit{where } \varphi(t) = \frac{1}{2}\big|\operatorname{sign} f(t) - \operatorname{sign} f'(t)\big|. \tag{2.2}$$

Proof. It is sufficient to prove this inequality for functions with simple zeros; the general case follows from the fact that a root of f of multiplicity $k > 1$ is also a root of f' of multiplicity $k-1$.

The common form of the Rolle theorem asserts that between any two roots of f there must be a root of f', which immediately gives the inequality $N(f) \leqslant N(f')+1$. But if, say, $\varphi(0) = 0$, then $f(0)$ and $f'(0)$ have the same sign, and therefore between 0 and the smallest root of f there must also be a root of f'. The same applies to the interval between the biggest root of f and 1. One can easily check that for all combinations of signs inequality (2.2) remains valid. □

Remark. Note that (2.2) can be viewed as a special case of the general inequality (1.5) for a segment that has zero curvature, and $2\pi\varphi(t) = \theta_t$, $t = 0, 1$.

Let now $\Pi_{\mathbf{p}} \subset \mathbb{R}^n$ be a hyperplane passing through the origin and orthogonal to the vector $\mathbf{p} \in \mathbb{S}^{n-1}$. We denote by $N(\mathbf{p}, \ell)$ (resp., $N(\mathbf{p}, \ell')$) the number of intersections of the curve ℓ (resp., ℓ') with $\Pi_{\mathbf{p}}$. The refined Rolle theorem (2.2) can be applied to the function $f(t) = \langle \mathbf{p}, \mathbf{x}(t)\rangle$ and its derivative $f'(t) = \langle \mathbf{p}, \dot{\mathbf{x}}(t)\rangle$ yielding the inequality

$$\forall \mathbf{p} \in \mathbb{S}^{n-1} \qquad N(\mathbf{p}, \ell) \leqslant N(\mathbf{p}, \ell') + \varphi_{\mathbf{p}}(0) - \varphi_{\mathbf{p}}(1), \tag{2.3}$$

where we set $\varphi_{\mathbf{p}}(t) = \frac{1}{2} \cdot \left|\operatorname{sign}\langle \mathbf{p}, \mathbf{x}(t)\rangle - \operatorname{sign}\langle \mathbf{p}, \dot{\mathbf{x}}(t)\rangle\right|$.

Now the assertion of Theorem 4 follows easily from the following principle well known from the integral geometry: for any smooth curve ℓ the length of its spherical projection $\mathbf{L}(\ell)$ is equal (modulo a factor π) to the average number of intersections of ℓ with the variable hyperplane $\Pi_{\mathbf{p}}$ [1]:

$$\mathbf{L}(\ell) = \frac{\pi}{\sigma_{n-1}} \cdot \int_{\mathbb{S}^{n-1}} N(\mathbf{p}, \ell)\, d\sigma_{n-1}(\mathbf{p}). \tag{2.4}$$

Averaging inequality (2.3) over all hyperplanes, we obtain

$$\mathbf{L}(\ell) \leqslant \mathbf{L}(\ell') + \frac{\pi}{\sigma_{n-1}} \biggl(\int_{\mathbb{S}^{n-1}} \varphi_{\mathbf{p}}(0)\, d\sigma_{n-1}(\mathbf{p}) - \int_{\mathbb{S}^{n-1}} \varphi_{\mathbf{p}}(1)\, d\sigma_{n-1}(\mathbf{p}) \biggr).$$

Note that for any t the integral

$$\int_{\mathbb{S}^{n-1}} \varphi_{\mathbf{p}}(t)\, d\sigma_{n-1}(\mathbf{p})$$

is the spherical measure of the hyperplanes $\Pi_{\mathbf{p}}$ separating the vector $\mathbf{x}(t)$ from the vector $\dot{\mathbf{x}}(t)$, i.e., intersecting the line segment $[\mathbf{x}(t), \dot{\mathbf{x}}(t)] \subset \mathbb{R}^n$. By the same integral geometric inequality (2.4), this measure is equal to the spherical distance $\operatorname{dist}_{\mathbb{S}^{n-1}}(\mathbf{x}(t), \dot{\mathbf{x}}(t))$ modulo the factor π/σ_{n-1}. □

Remark on the singular case. Inequality (2.1) was established under some genericity and regularity assumptions. In fact, they can be dropped when the definitions are properly modified. If ℓ is only piecewise C^2-smooth, then the velocity curve may have jumps, being only a piecewise continuous C^1-curve. In this case one should set $\mathbf{L}(\ell')$ equal to the sum of lengths of connected components of the spherical projection of ℓ' plus the total *spherical* length of all jumps of this projection.

Next, if ℓ passes through the origin, then its spherical projection will also have jumps which must be incorporated into the length of the spherical projection (e.g., if $\mathbf{x}(t_0) = 0 \in \mathbb{R}^n$ and $\mathbf{x}(t)$ is C^2-smooth near t_0, then the spherical length of the jump at t_0 is equal to π since the point instantly moves to the central symmetric position).

All these refinements can be justified by analyzing small perturbations and/or smoothing the curves. We leave the details to the reader. Note also that inequality (2.2), which is essentially the classical Rolle theorem, can be regarded as a special case of (2.1): for the scalar function $f(t)$ inequality (2.1), applied to the plane curve $t \mapsto (f(t), \varepsilon t)$ with small $\varepsilon > 0$, yields (2.2) as a limit case.

2.3. Pursuit on Riemannian manifolds. In this section we give an alternative proof of (2.1). It is based on the following construction. Let M be an arbitrary complete Riemannian manifold and $\widetilde{\ell}, \widetilde{\ell}'$ $[0,1] \to M$ be two parametrized curves. We say that the curve $\widetilde{\ell}$ *pursues* the other curve $\widetilde{\ell}'$ if, for almost all moments of time $t \in [0,1]$, the velocity vector of $\widetilde{\ell}$ points towards the current point of $\widetilde{\ell}'$. In other words, the velocity vector is tangent to the shortest geodesic curve connecting $\widetilde{\ell}(t)$ with $\widetilde{\ell}'(t)$ in the "right" direction.

Remark. This definition also applies to the case where the "pursued" curve is only piecewise continuous. If, at a certain moment, the two curves occupy cutting points, then the choice of the geodesics from the *global* minimizers is not specified.

Lemma 2 (integral triangle inequality, *alias* pursuit principle). *Suppose that one rectifiable curve $\widetilde{\ell}$ pursues another rectifiable curve $\widetilde{\ell}'$ on a complete Riemannian manifold.*

Then their lengths $\mathbf{L}(\widetilde{\ell})$ and $\mathbf{L}(\widetilde{\ell}')$ are related by the inequality $\mathbf{L}(\widetilde{\ell}) \leqslant \mathbf{L}(\widetilde{\ell}') + \Phi(0) - \Phi(1)$, where $\Phi(t) = \operatorname{dist}_M(\widetilde{\ell}(t), \widetilde{\ell}'(t))$ is the Riemannian distance between the points on the curves at the moment $t \in [0,1]$.

Proof. We prove this inequality assuming that $\widetilde{\ell}(t)$ and $\widetilde{\ell}'(t)$ occupy cutting and focal positions only at discrete moments of time, so that the distance $\Phi(t)$ is piecewise-differentiable. At the moment t of uniqueness we take a geodesic curve γ_t connecting $\widetilde{\ell}(t)$ with $\widetilde{\ell}'(t)$. Then $\Phi(t)$ is the length of the segment of γ_t between the two points. Obviously, $\dot{\Phi}(t) = p(\mathbf{v}'(t)) - p(\mathbf{v}(t)) \leqslant \|\mathbf{v}'(t)\| - \|\mathbf{v}(t)\|$, where $p(\cdot) \in \mathbb{R}$ denotes the projection of a vector onto the tangent direction to γ_t (taken with the natural sign), and $\|\cdot\|$ is the length of the vector. Indeed, $p(\mathbf{v}(t)) = \|\mathbf{v}(t)\|$ by the pursuit condition, and the second projection does not exceed the norm $\|\mathbf{v}'(t)\|$. Integrating this inequality from 0 to 1 and observing that $\mathbf{L}(\widetilde{\ell}) = \int_0^1 \|\mathbf{v}(t)\|\,dt$, we obtain $\Phi(1) - \Phi(0) \leqslant \mathbf{L}(\widetilde{\ell}') - \mathbf{L}(\widetilde{\ell})$, which is what we need.

If M is a sphere, then the regularity assumption about focal points is, in fact, obsolete since on the "stalemate" subset of $[0,1]$ the curves $\widetilde{\ell}$ and $\widetilde{\ell}'$ are symmetric, and, hence the corresponding points produce equal paths. We will not discuss the general case. □

Remark. Apparently, Lemma 2 can be proved in a more general setting. Let (M, ρ) be a metric space and consider two discrete time trajectories $x_1, \dots, x_n, y_1, \dots, y_n$, $x_i, y_i \in M$. We say that the trajectory $\{y_i\}$ pursues $\{x_i\}$ if, at any moment $i = 1, \dots, n-1$, the point y_{i+1} lies on the "segment" connecting y_i and x_{i+1}, i.e., $\rho(y_i, y_{i+1}) + \rho(y_{i+1}, x_{i+1}) = \rho(y_i, x_{i+1})$ (the triangle degenerates into equality). A discrete analog of Lemma 2 asserts that in this case the (metric) lengths $\mathbf{L}_y = \sum_{i=1}^{n-1}\rho(y_i, y_{i+1})$ and $\mathbf{L}_x = \sum_{i=1}^{n-1}\rho(x_i, x_{i+1})$ are related by the inequality

$$\mathbf{L}_y \leqslant \mathbf{L}_x + \rho(x_n, y_n) - \rho(x_1, y_1).$$

This is the result of addition of several triangle inequalities.

Lemma 3. *If $\ell\ [0,1] \to \mathbb{R}^n$, $t \mapsto \mathbf{x}(t)$, is a piecewise smooth curve, then its spherical indicatrix $\widetilde{\ell}$ (the spherical projection of ℓ onto the unit sphere $\mathbb{S}^{n-1}$) pursues the indicatrix $\widetilde{\ell'}$ of the derived curve $\ell'\ t \mapsto \dot{\mathbf{x}}(t)$.*

Proof. The velocity vector of $\widetilde{\ell}$ always lies in the 2-dimensional plane spanned by $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$ (a tautology!), and, hence the situation is in fact planar: the curve $\widetilde{\ell}$ is tangent to the large circle passing through $\widetilde{\ell}(t)$ and $\widetilde{\ell'}(t)$ at any moment t. One needs only to check that the direction (one of the two possible directions along the circle) is correct, which is completely obvious in the plane case: if the oriented angle from $\widetilde{\ell}(t)$ to $\widetilde{\ell'}(t)$ is positive, then the vector $\widetilde{\ell}(t)$ itself rotates in the positive direction and vice versa. □

Lemma 2 and Lemma 3 taken together imply inequality (2.1), thus completing the second proof of Theorem 4. □

2.4. Rolle theorems for analytic functions. We shall prove now inequality (1.6) which is the Rolle theorem for analytic functions.

Let $f(z)$ be a function of one complex variable $z \in \mathbb{C}$, analytic in some neighborhood of the curve $\Gamma \subset \mathbb{C}$ parametrized as $[0,1] \ni t \mapsto z(t)$. In this case the function $f\big(z(t)\big)$ restricted on $[0,1]$ can be regarded as a parametrization of the image curve $\ell = f(\Gamma)$ in $\mathbb{R}^2 \simeq \mathbb{C}^1$. The spherical projection can be identified with the argument $\operatorname{Arg} f(z(t))$ or, more precisely, with the function $t \mapsto \exp\big(i \operatorname{Arg} f(z(t))\big)$. The (angular) length of the spherical projection is the *total variation of argument* of the function f along Γ, and thus $\mathbf{L}(\ell) = 2\pi \mathbf{V}_\Gamma(f)$. On the other hand, the derived curve ℓ' in this case is $t \mapsto f'(z(t)) \cdot \dot{z}(t)$, where f' is the complex derivative of f. Finally, using (1.4) and the triangle inequality (1.8), we establish (1.5) since

$$\mathbf{L}(\ell') \leqslant 2\pi\mathbf{V}_\Gamma(f') + 2\pi\mathbf{V}_\Gamma(\dot{z}) = \mathbf{V}_\Gamma(f') + \varkappa(\Gamma), \qquad \theta_t = \Phi(t) \text{ for } t = 0, 1.$$

The other inequalities, (1.6) and (1.7), are obvious corollaries of (1.5). □

Remark about the Voorhoeve inequality. Inequality (1.5) applied to a complex-valued function on the real interval $[0,1]$ implies the Voorhoeve inequality (1.2). Indeed, the curvature of the interval is zero and both θ_t, $t = 0, 1$, are between 0 and π. Hence their difference is at most π, and after division by 2π one obtains the term $\dfrac{1}{2}$ in (1.2). Alternatively, we could consider a closed loop squeezed onto a segment as $t \mapsto 2t$ for $t \in \left[0, \frac{1}{2}\right]$ and $t \mapsto 2(1-t)$ for $t \in \left[\frac{1}{2}, 1\right]$. The total curvature of such a singular loop is 2π (the sum of two rotation angles at endpoints, each one being π), the total variation of f and f' will be doubled, $\mathbf{V}_{[0,1]}(f, f') = \frac{1}{2}\mathbf{V}_\Gamma(f, f')$, and, hence we again arrive at inequality (1.2), this time starting from the closed-loop form (1.6).

2.5. Lucas theorem. We can now easily estimate the Voorhoeve index of a rational function. For the C^2-smooth closed curve $\ell \subset \mathbb{C}$ and the point $a \notin \ell$ the absolute rotation $R_a(\ell)$ does not exceed (by Corollary 1 to Theorem 4) the integral curvature of ℓ. Using (1.8), one easily obtains an upper bound for rational functions.

Proposition. *If $f(z) = p(z)/q(z)$ is a rational function, $p, q \in \mathbb{C}[z]$, then*

$$\mathbf{V}_\Gamma(f) \leqslant \frac{\varkappa(\Gamma)}{2\pi}(\deg p + \deg q). \quad \square \tag{2.5}$$

Remark on zeros of polynomials. If ℓ is a closed *convex* curve containing all roots of the polynomial p, then the argument $\operatorname{Arg} p$ is monotonous along ℓ, and, hence $\mathbf{V}_\ell(p) = \deg p$. On the other hand, $\mathbf{V}_\ell(p') \leqslant \deg p' = \deg p - 1$, and the equality can occur only if all zeros of p' also lie inside ℓ. Meanwhile, the complex Rolle theorem asserts that $p = \mathbf{V}_\ell(p) \leqslant \mathbf{V}_\ell(p') + 1 \leqslant p - 1 + 1$. Hence, all inequalities must be equalities in this case and we have re-established the well-known fact (known as the Lucas theorem) that all roots of the derivative p' belong to the convex hull of the roots of p.

3. Applications to quasipolynomials

3.1. Geometric inequalities. Let $\Gamma \subset \mathbb{C}$ be a rectifiable plane curve and $f(z) = e^{\mu z}$. We estimate $\mathbf{V}_\Gamma(f)$ for two important special cases:

(a) Γ is a closed curve bounding a *convex* domain $U \subset \mathbb{C}$,
(b) Γ is a line segment of finite length $\mathbf{l}(\Gamma)$.

In the first case we have

$$\begin{aligned} \mathbf{V}_\Gamma(f) &= \frac{1}{2\pi}\int_\Gamma \left|\operatorname{Im}(f'(z)/f(z)\,dz)\right| = \frac{1}{2\pi}\int_\Gamma \left|\operatorname{Im}(\mu\,dz)\right| = \\ &= \frac{1}{2\pi}|\mu| \times (\text{total length of projection of } \Gamma \text{ on the line } \mathbb{R}i\mu) = \\ &= \frac{1}{2\pi}|\mu| \times 2(\text{length of projection of } U \text{ on this line}) = \\ &= \frac{1}{\pi}\|\mu\|_U \leqslant \qquad (\text{see } 1.4) \\ &\leqslant |\mu|\frac{\operatorname{diam}_{\mathbb{C}} U}{\pi}. \end{aligned} \tag{3.1}$$

In the second case, one can obviously write

$$\mathbf{V}_\Gamma(f) = \frac{1}{\mathrm{l}(\Gamma)} 2\pi \times (\text{length of projection of } \mu \text{ on the direction parallel to } \Gamma). \tag{3.2}$$

We will also need an upper estimate for the length of a polyline passing through all points of the finite set $\Lambda \Subset \mathbb{C}$ in terms of the Euclidean diameter of the latter. In the best case, where Λ is a finite subset of a rectifiable plane curve C, we have $\mathcal{L}(\Lambda) \leqslant \mathrm{l}(C)$.

Lemma 4. *If Λ is a finite subset of the unit square*

$$\{0 < \operatorname{Re} z < 1,\ 0 < \operatorname{Im} z < 1\},$$

then $\mathcal{L}(\Lambda) \leqslant 2(\sqrt{\#\Lambda} + 1)$.

Proof. We denote $n = \#\Lambda$ and let m be the integer part of $\sqrt{\#\Lambda} + 1$. We break the square into m horizontal strips of height m^{-1}, and in each even (resp., odd) strip find a polyline starting in the upper left (upper right) corner, visiting all points of this strip in the order of increase (decrease) of their abscissas, and ending in the lower left (resp., lower right) corner. Then the length of each of these polylines is at most $1 + (n_i + 1)/m$, where n_i is the number of points in the ith strip. Clearly, all these polylines can be joined into a common polyline visiting all points of Λ. The overall length of the polyline thus constructed will be at most

$$\sum_i \left[1 + m^{-1}(n_i + 1)\right] \leqslant m + n/m + 1 \leqslant \sqrt{n} + 1 + \sqrt{n} + 1. \qquad \square$$

Corollary. $\mathcal{L}(\Lambda) \leqslant 2(\sqrt{\#\Lambda} + 1) \cdot \operatorname{diam}_{\mathbb{C}}(\Lambda)$. $\square$

Remark. The uniform square grid of m^2 points inside a unit square shows that the length of the shortest polyline cannot be made less than m since the distance between any two points is $1/m$. In fact, the numerical proportionality coefficient 2 can be slightly improved by taking a symmetric hexagon of diameter $\operatorname{diam}_{\mathbb{C}}(\Lambda)$ to cover Λ instead of the square. On the other hand, the hexagonal grid gives a slightly better lower bound.

3.2. Upper bounds for the number of complex zeros of a quasipolynomial. In this section we prove Theorem 2. The quasipolynomial $f = \sum_j e^{\mu_j z} p_j(z)$ with the spectrum $\Lambda = \{\mu_1, \dots, \mu_k\}$, $\mu_j \neq \mu_k$ for $j \neq k$, solves the linear equation

$$D^{n_k - 1} e^{(\mu_{k-1} - \mu_k) z} D^{n_{k-1}} e^{(\mu_{k-2} - \mu_{k-1}) z} \dots D^{n_2} e^{(\mu_1 - \mu_2) z} D^{n_1} e^{-\mu_1 z} f = \text{const}, \tag{3.3}$$

where $n_j = \deg p_{\mu_j} + 1$. Without loss of generality, one can assume that $\mu_1 = 0$ (otherwise we just divide f by $e^{\mu_1 z}$ without changing the number of zeros). Here $D : f \mapsto f'$ is the differentiation operator, and the degree of the above expression in D is $d - 1$, where d is the degree of the quasipolynomial.

By the complex Rolle theorem applied to the closed curve Γ, the differential equation (3.3), together with inequality (3.1), implies the inequality

$$\begin{aligned} \mathbf{V}_\Gamma(f) &\leqslant \mathbf{V}_\Gamma(\text{const}) + (d-1)\frac{\varkappa(\Gamma)}{2\pi} + \sum_{j=1}^{k-1} \mathbf{V}_\Gamma\left(e^{(\mu_j - \mu_{j+1})z}\right) \\ &\leqslant 0 + (d-1) + \frac{1}{\pi}\sum_j \|\mu_j - \mu_{j+1}\|_U = (d-1) + \frac{1}{\pi}\mathcal{L}_U(\Lambda) \\ &\leqslant (d-1) + \frac{1}{\pi}\operatorname{diam}_{\mathbb{C}} U \cdot \mathcal{L}(\Lambda), \end{aligned}$$

since $\varkappa(\Gamma) = 2\pi$ for the convex curve. We can choose the ordering of points on Λ following the shortest polyline, and since the number of zeros of f is at most $\mathbf{V}_\Gamma(f)$, we arrive then at the inequality asserted by Theorem 2; inequality (1.10c) follows from Lemma 4 and Corollary thereof. □

3.3. Comparison with previously known estimates. The estimate given by Theorem 2 is quite accurate if we consider Λ to be fixed and let the degrees grow to infinity: the Chebyshev excess remains bounded and proportional to the size of U. This boundedness of s will persist if we consider a specific case of quasipolynomials with essentially one-dimensional spectrum (say, an arbitrary finite subset of a fixed smooth curve). Moreover, if the spectrum Λ is real and U is a real line, then, obviously, $\mathcal{L}_U(\Lambda) = 0$, so that $s(d, \Lambda, U) = 0$ and we arrive at the well-known Chebyshev property or real quasipolynomials.

Voorhoeve proved in [7], using inequality (1.1), that the number of zeros of a quasipolynomial of degree d in a circular disk U can be at most $2(d -$

$1) + \frac{2}{\pi}\operatorname{diam}_{\mathbb{C}}(\Lambda) \cdot \operatorname{diam}_{\mathbb{C}}(U)$; this result overrides previous achievements by Tijdeman and Waldschmidt (see [7] for references). The inequality from [7] asserts that

$$s(d, \Lambda, U) \leqslant (d-1) + \frac{1}{\pi}\operatorname*{diam}_{\mathbb{C}} U \cdot \operatorname*{diam}_{\mathbb{C}} \Lambda = d + O(1). \qquad (3.4)$$

This estimate is worse than (()1.10) for large d, though it can be better for small d and relatively large domains U.

3.4. Discussion. 1. As follows from the above inequalities, the worst case from the point of view of counting the number of zeros of quasipolynomials is the case where f is a linear combination of exponentials with points of spectrum filling more or less uniformly a domain in $\mathbb{C}$ with nonempty interior. It is not clear for the moment whether the Chebyshev excess indeed increases to infinity as $\#\Lambda \to \infty$.

2. An alternative form of the question concerning the number of zeros of quasipolynomials is to ask about the size of domains (in particular, circular disks) in which the quasipolynomials possess the Chebyshev property (iė; $s = 0$). The above inequalities can be transformed into results of this type by inverting the inequality $s < 1$ (since then the number of zeros, being integer, must be at most $d-1$).

3. The above results can also be easily generalized from quasipolynomials to the class of the so-called generalized quasipolynomials that solve differential equations of the form

$$\left(\prod_{j=1}^{d+1} e^{\mu_j z} R_j(z)\, D\right) f = 0, \qquad D = \frac{d}{dz},$$

where R_j are rational functions of known degrees. Since we have $\mathbf{V}_\Gamma(R_j) \leqslant (\deg p_j + \deg q_j) \cdot \frac{\varkappa(\Gamma)}{2\pi}$ for the rational function $R_j = p_j(z)/q_j(z)$ by virtue of (2.5), all the above constructions can be easily repeated in this case.

4. The Bernstein index and the equivalence of two Rolle theories

In this section we recall briefly some elementary properties of the Bernstein index $f \mapsto \mathbf{B}_{K,U}(f)$ defined in (1.3) and prove Theorem 3. For more details on the Bernstein index see [2], [6].

4.1. Geometry of plane sets. Let $U \subset \mathbb{C}$ be a simply connected plane domain conformally equivalent to the unit open disk $\mathbb{D} = \{z : |z| < 1\}$. As such, it inherits the hyperbolic metric $\rho_U(\cdot, \cdot)$ invariant by conformal automorphisms of U. We write $K \Subset U$ if the *hyperbolic diameter* $\operatorname{diam}_U(K) = \sup_{z,w \in K} \rho_U(z, w)$ of the subset $K \Subset U$ is finite (the symbol $\operatorname{diam}_{\mathbb{C}}$ stands, as before, for the Euclidean diameter). By $\mathrm{l}(\Gamma)$ we will denote the Euclidean length of a rectifiable (usually even piecewise smooth) curve $\Gamma \subset \mathbb{C}$. If $K \Subset U$, then the Euclidean distance from K to the complement $\mathbb{C} \smallsetminus U$ must be positive, and it will be denoted by $\theta(K, U)$ (the thickness of the gap between K and U). Next, for any bounded subset $K \subset \mathbb{C}$ there exists at least one covering of K by a finite union of Euclidean disks with diameters d_j; we denote by $\eta(K) = \inf_{\mathfrak{C}(K)} \sum_j d_j$ the infimum taken over all these finite coverings of K by disks. If K contains a line segment of length $r > 0$, then, obviously, $\eta(K) \geqslant r > 0$.

There are several inequalities relating these geometric characteristics. For example, the Köbe $\frac{1}{4}$-theorem implies that $\operatorname{diam}_U(K) \leqslant \dfrac{\mathrm{l}(\partial U)}{\theta(K, U)}$ for the convex domain U. In any case, all of the above characteristics can be easily computed for simple domains (or pairs of domains) in $\mathbb{C}$.

In what follows we denote by $\mathcal{A}(U)$ the space of functions holomorphic on the closure of the domain U; the domain itself is always assumed to be bounded by a sufficiently regular curve ∂U (we simply say a "regular domain"). We also denote $\ln_+ a = \max(\ln a, 0)$.

4.2. Basic properties of the Bernstein index. For a pair of nested sets $K \Subset U$ and the function $f \in \mathcal{A}(U)$ the Bernstein index $\mathbf{B}_{K,U}(f) \geqslant 0$ was defined by (1.3). The relation $\mathbf{B}_{K,U}(f) = 0$ implies that $f = \text{const}$, and we set $\mathbf{B}_{K,U}(0) = 0$ by definition.

This index possesses a number of useful properties relating it with the degree of a polynomial (or the number of zeros of an analytic function). Among these properties, we will need the following ones.

1. Bernstein inequality. If $p \in \mathbb{C}[z]$ is a polynomial of degree n, $K = [-1, 1] \Subset \mathbb{C}$ is a segment, and $U = U_R = \{|t-1| + |t+1| \leqslant 2R\}$ is a confocal ellipse, then $B_{K,U}(p) \leqslant \gamma_R\, d$, where $\gamma_R = \ln R$ (S. Bernstein, 1926). Moreover, one can easily show that for any pair $K \Subset U$ with $\eta(K) > 0$ there exists a finite positive geometric constant $\gamma_0 = \gamma_0(K, U)$ such that

$$\mathbf{B}_{K,U}(p) \leqslant \gamma_0\, n. \tag{4.1}$$

One can take $\gamma_0 = \ln_+\big(eD/\theta(K, U)\eta(K)\big)$, where we denote $D = \operatorname{diam}_{\mathbb{C}}(U)$, and $e = 2.718\ldots$ is the Euler number.

2. Generalized Jensen inequality. The number of isolated zeros of an analytic function can be estimated from above via its Bernstein index [2]:

if $f \not\equiv 0$, then there exists $\gamma_1 = \gamma_1(K, U)$ such that for the number $N_K(f)$ of isolated zeros of f in K we have

$$N_K(f) \leqslant \gamma_1 \mathbf{B}_{K,U}(f), \qquad \gamma_1 = \frac{1}{2}(e^\delta + 1). \tag{4.2}$$

Here $\delta = \delta(U, K) = \operatorname{diam}_U(K)$.

3. **Rolle theorem for analytic functions.** The following inequality holds for any $f \in \mathcal{A}(U)$ [6]: if $K^- \Subset K \Subset U$, then there exists a finite geometric constant $c = c(K^-, K, U)$ such that

$$\mathbf{B}_{K,U}(f) \leqslant \mathbf{B}_{K^-,U}(f') + c, \qquad c = 1 + \ln_+ \left(\frac{\mathrm{l}(\partial K) \cdot \mathrm{l}(\partial U)}{\theta^2(K^-, K)} \right). \tag{4.3}$$

4. **Weak dependence on the inner set.** The Bernstein index is a zero-order homogeneous functional on $\mathcal{A}(U)$ whose equivalence class does not depend on the inner set K as soon as $\eta(K) > 0$ [6]: if $\lambda \neq 0$, then $\mathbf{B}_{K,U}(\lambda f) = \mathbf{B}_{K,U}(f)$, and if $\eta(K_1) \cdot \eta(K_2) > 0$, then there exists a positive finite $\gamma_2 = \gamma_2(K_1, K_2, U)$ such that

$$\forall f \in \mathcal{A}(U) \qquad \frac{1}{\gamma_2} \mathbf{B}_{K_1,U}(f) \leqslant \mathbf{B}_{K_2,U}(f) \leqslant \gamma_2 \mathbf{B}_{K_1,U}(f). \tag{4.4}$$

5. **Almost additivity.** The Bernstein index is almost additive in the following weakened sense: the index of the product fg of two functions cannot exceed too much the sum of the indices of f and g and must not be much smaller than the absolute value of their difference [6]. In particular, if $p \in \mathbb{C}[z]$ is a polynomial, and f is an analytic function without zeros in the closure of U, and $K \Subset U^- \Subset U$, then

$$\mathbf{B}_{K,U}(pf) \leqslant (1 + e^{2\delta}) \mathbf{B}_{K,U}(f) + \gamma_3 n, \quad \gamma_3 = \ln_+ \big(eD/\eta(K)\big), \tag{4.5}$$

$$\mathbf{B}_{K,U^-}(f) \leqslant \mathbf{B}_{K,U}(pf) + \gamma_4 n, \qquad \gamma_4 = \ln_+ \big(eD/\theta(U^-, U)\big), \tag{4.6}$$

where $n = \deg p$, $\delta = \operatorname{diam}_U(K)$, $D = \operatorname{diam}_{\mathbb{C}}(U)$, and γ_3, γ_4 are geometric constants. It is this simplest form that we will need (compare (4.5), (4.6) with inequalities (1.8), (1.9)). The general case is discussed in [6].

4.3. Bernstein index and linear differential equations. Relations (4.2)–(4.6) are less convenient than the corresponding inequalities for the Voorhoeve index. However, the Bernstein index can be easily computed (or rather estimated from above) for an analytic function solving a linear differential equation of the form

$$y^{(n)} + a_1(z)\, y^{(n-1)} + \cdots + a_{n-2}(z)\, y'' + a_{n-1}(z)\, y' + a_n(z)\, y = 0$$

in terms of the maximum modulus of the coefficients $a_k(z)$, even if the coefficients a_k are variable. Indeed, the above equation is equivalent to the system of first-order linear equations

$$\dot{\mathbf{Y}}(t) = A(t)\mathbf{Y}(t), \qquad \mathbf{Y} = (y, y', y'', \dots, y^{(n-1)}) \in \mathbb{C}^n,$$

where $A(t)$ is a matrix-valued analytic function in the companion form with the bounded norm $\|A(t)\| \leqslant A$ everywhere in U. The growth of solutions of a linear system can be easily controlled: $\max_{t\in U} \|\mathbf{Y}(t)\| \leqslant \exp(\gamma A) \cdot \max_{t\in K} \|\mathbf{Y}(t)\|$, where γ is determined by the geometry of the pair $K \Subset U$. But this means that for *some* component of the vector $\mathbf{Y}$ a similar inequality holds, and, hence the Bernstein index of some derivative $y^{(k)}$, $0 \leqslant k \leqslant n-1$, admits an upper bound in terms of the magnitude of the coefficients $a_j(t)$ in U. Iterating the Rolle theorem in form (4.3), one obtains a similar estimate for the solution $y(t)$ itself. As a consequence, we infer that if f satisfies the above equation with $\max_{z\in U} |a_k(z)| \leqslant A$ for all $k = 1, 2, \dots, n$, then the number of zeros of f in any $K \Subset U$ can be as large as $O(A + n \ln n)$ modulo a factor dependent on the geometry of the pair K, U (in [2] this is proved for the case where K is a real segment and f is real on K, and in [6] the general complex case is considered).

Still the question of how many really independent Rolle theories do coexist is quite natural. The Riemann mapping theorem immediately implies that knowing the Bernstein index of an analytic function does not impose any upper bound on its Voorhoeve index, even if the function has no zeros; the inverse is also true. However, Theorem 3 asserts a natural equivalence of the two constructions in a weaker sense, after diminishing the domains. The proof of this result occupies 4.4–4.6.

4.4. Nonexpansion principle. We start with the following reformulation of the classical Schwartz–Pick lemma; see [4]. Let $f\, U \to \mathbb{C}$ be an analytic function defined in the hyperbolic domain U. Assume that $f(U) \subseteq V$, where V is yet another hyperbolic domain.

Lemma 5 (Nonexpansion principle [4], p. 14). *The map f is nonexpanding in the hyperbolic metrics in U and V:*

$$\forall z, w \in U \qquad \rho_V\big(f(z), f(w)\big) \leqslant \rho_U(z, w),$$

and the equality is possible if and only if f is a conformal isomorphism between U and V.

Proof. Obviously, it is sufficient to prove this property for an automorphism of the unit disk $f\, \mathbb{D} \to \mathbb{D}$, having a fixed point at the origin, $f(0) = 0$. But in this case the assertion coincides with the Schwartz lemma: since $\rho_{\mathbb{D}}(0, z)$ is a monotonic function of $|z|$, we have

$$|f(z)| \leqslant |z| \implies \rho_{\mathbb{D}}\big(f(z), 0\big) \leqslant \rho_{\mathbb{D}}(z, 0). \qquad \square$$

Corollary. *Let f be analytic in U and take values in the right half-plane, $f(U) \subset \mathbb{H} = \{\operatorname{Re} z > 0\}$. Assume that $f(a) = 1$ for some point $a \in U$.*

Then for any set $K \Subset U$ the image of K belongs to the Euclidean disk $D_\delta = \{w \in \mathbb{H} : |w - \cosh\delta| \leqslant \sinh\delta\}$, where $\delta = \operatorname{diam}_U(K \cup a) < \infty$.

Indeed, in the hyperbolic metric $(\operatorname{Re} w)^{-1}|dw|$ the set D_δ is also the disk of the hyperbolic diameter δ centered at $w = 1$: $D_\delta = \{w \in \mathbb{H}\ \rho_{\mathbb{H}}(1, w) < \delta\}$. □

4.5. Demonstration of Theorem 3 for functions without zeros. If a function f has no zeros in some domain, then a branch of logarithm $\ln f$ or the power f^λ can be selected.

Estimating $\mathbf{B}$ through $\mathbf{V}$. Let $K \Subset U$ be a nested pair of sets and suppose that a simple closed path $\Gamma = \partial U'$ encircles the domain U remaining at a positive distance from it, so that $K \Subset U \Subset U'$. We denote by $\delta = \operatorname{diam}_\Gamma(U) < \infty$ the hyperbolic diameter of the set U with respect to U'.

Suppose that f has no zeros on $\Gamma \cup U'$ and denote $\mathbf{V}_\Gamma(f) = v$. Let $z_0 \in K$ be the point where the maximum of $f(z)$ on K is achieved. Without loss of generality we can assume that $f(z_0) = 1$. Consider $g(z) = f(z)^{1/4v}$. Then g takes Γ into a curve entirely belonging to the right half-plane $\mathbb{H}$ (since the total variation of $\operatorname{Arg} g(z)$ on Γ is no greater than $\pi/2$).

By the nonexpansion principle, the domain U of a finite hyperbolic diameter δ (in U') is taken into the circular disk D_δ of the Euclidean diameter $e^\delta - e^{-\delta} = 2\sinh\delta$. Note that $\max\{|w|\, w \in D_\delta\} \leqslant e^\delta$, and hence $\max_U |g(z)| \leqslant e^\delta$ and, finally, $\max_U |f(z)| \leqslant e^{4\delta v}$. Thus $\mathbf{B}_{K,U}(f) \leqslant 4\delta v$, where $v = \mathbf{V}_\Gamma(f)$, and (1.12) is proved for f without zeros. Note also that this estimate is valid for any choice of the subset K, even for a single point in U, but in order to achieve (1.12) for the general case, we will need $\eta(K) > 0$.

Estimating $\mathbf{V}$ through $\mathbf{B}$. Conversely, let $K \Subset U$ and Γ be a simple closed path strictly inside U: $\Gamma \Subset U$. We denote $\mathbf{B}_{K,U}(f) = b$. The function $g(z) = \dfrac{1}{b} \operatorname{Ln} f(z)$ is well defined and maps U into the translated right half-plane $\mathbb{H}' = \{\operatorname{Re} w \leqslant 1\}$. Without loss of generality we assume that $g(z_0) = 0$ for $z_0 \in K$. For the closed curve $\Gamma \Subset U$ we have

$$\begin{aligned}\mathbf{V}_\Gamma(f) &= b \times (\text{length of projection of } g(\Gamma) \text{ onto the imaginary axis})\\ &\leqslant b \cdot \mathrm{l}(g(\Gamma)) \leqslant b \cdot \mathrm{l}(\Gamma) \cdot \max_{z \in \Gamma} |g'(z)|.\end{aligned}$$

Choose a subset $U' \Subset U$ containing Γ strictly inside. By the Cauchy integral formula,

$$\max_\Gamma |g'(z)| \leqslant \frac{\mathrm{l}(\partial U')}{2\pi\theta(\Gamma, U')} \max_{U'} |g(z)|.$$

By the nonexpansion principle, the latter maximum does not exceed e^{δ}, where $\delta = \operatorname{diam}_U(U' \cup K)$, the whole expression can thus be regarded as a geometric constant $\gamma_5 = e^{\delta}\dfrac{\mathrm{l}(\partial U')}{2\pi\theta(\Gamma, U')}$, and we arrive at the inequality $v \leqslant \gamma_5 b$ which is the special case of (1.11) for functions without zeros. □

4.6. The general case of Theorem 3. The general case is reduced to the case of nonvanishing functions by inequalities (1.8), (1.9), (4.5), (4.6) after representing the arbitrary function f in the form $f = Fp$, where p is a unitary polynomial and F has no zeros.

If $f \in \mathcal{A}(U)$ and $\mathbf{B}_{K,U}(f) = b$ for some $K \Subset U$, then for an arbitrary set U' such that $K \Subset U' \Subset U$ and $\Gamma \Subset U'$ the number of zeros of f in U' can be at most $d \leqslant \gamma_6 b$ by (3.2) and (3.4), where γ_6 is a geometric constant. Let p be a unitary polynomial of degree d with roots at these zeros. The function $F = f/p$ will be analytic and nonvanishing in U', and by (4.5) its Bernstein index with respect to the pair K, U' admits an upper estimate $\mathbf{B}_{K,U'}(f/p) \leqslant b + \gamma_4 d \leqslant b(1 + \gamma_4\gamma_6)$. Then inequality (1.11) already proved for this case implies that $\mathbf{V}_\Gamma(f/p) \leqslant \gamma_7\, b$, and, hence $\mathbf{V}_\Gamma(f) \leqslant \gamma_7 b + d \cdot \varkappa(\Gamma)/2\pi = \gamma_8 b$.

The chain of inequalities that proves the inverse sense inequality is completely similar: if p is a unitary polynomial with the same zeros as f inside Γ, then $d = \deg p \leqslant v$, and by (2.5) $\mathbf{V}_\Gamma(f/p) \leqslant v\,(1 + \varkappa(\Gamma)/2\pi) = \gamma_9 v$. Now (1.12) can be applied to the invertible function f/p, yielding the inequalities

$$\mathbf{V}_\Gamma(f/p) \leqslant \gamma_9 v \overset{(1.12)}{\Longrightarrow} \mathbf{B}_{K,U}(f/p) \leqslant \gamma_{10}\, v \overset{(4.5)}{\Longrightarrow} \mathbf{B}_{K,U}(f) \leqslant \gamma_{11}\, v.$$

All constants γ_i that appear above are geometric, and thus the assertion of Theorem 3 is established in full generality. □

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