§27. Addendum 3. Algebra and Mixed Volumes (A. G. Khovanskiĭ)

Here we discuss the recently discovered intimate relationship between the theory of mixed volumes and algebra. This relationship enables one, in particular, to give an algebraic proof of the Alexandrov-Fenchel inequality, supplementing the ones known earlier.

27.1. Outline of the Algebraic Proof of the Alexandrov-Fenchel Inequality. We first describe the outline of the proof; in subsections 27.11–27.13 the proof is developed in detail.

27.1.1. To every *monomial* of the form $cz_1^{m_1}\dots z_n^{m_n}$ in n complex variables $z_1,\dots,z_n$ with complex coefficient c we relate the point with integer coordinates $m=(m_1,\dots,m_n)$ in the space $\mathbb{R}^n$. It may be viewed as the power of the monomial. By a *Laurent polynomial* we mean a finite sum of monomials in which the integers m_i may be both positive and negative. By the *Newton polyhedron* of a Laurent polynomial we mean the convex envelope in $\mathbb{R}^n$ of all the points with integer coordinates corresponding to all the monomials which appear in this polynomial with non-zero coefficients.

In this subsection we will agree to consider only polynomials with non-zero constant term. The Newton polyhedra of such polynomials contain the origin. The relationship between algebra and the theory of mixed volumes is established by the following theorem concerning the number of roots [BerD].

27.1.2. Theorem (on the number of roots). *The number of complex roots of the general (typical) system of polynomial equations $P_1=0,\dots,P_n=0$ with fixed Newton polyhedrons $\Delta_1,\dots,\Delta_n$ is equal to the mixed volume of the Newton polyhedra multiplied by $n!$.*

27.1.3. Remark. It is not necessary to assume that all the polynomials P_i have a non-zero constant term. The theorem will remain valid in the general case if, instead of the number of roots, we compute the number of complex roots all of whose coordinates are non-zero.

27.1.4. The key role in the sequel is played by the Hodge inequality, well in algebraic geometry. Here is the statement of this inequality. Suppose Γ_1 and Γ_2 are two complex curves contained in a compact complex algebraic surface F. Suppose the *index of self-intersection*[6] of one of these curves is positive. Then we have the following Hodge inequality

$$\langle\Gamma_1,\Gamma_2\rangle^2\geqslant\langle\Gamma_1,\Gamma_1\rangle\langle\Gamma_2,\Gamma_2\rangle,$$

where $\langle\Gamma_1,\Gamma_2\rangle$ is *the intersection index*[6] of the curves Γ_1 and Γ_2, while $\langle\Gamma_1,\Gamma_1\rangle$ and $\langle\Gamma_2,\Gamma_2\rangle$ are the self-intersection indices of these curves. Recall that the *index of self-intersection* of the curve is defined as the index of intersection of the curve

[6] See 27.11.4–27.11.7.

with its second copy obtained by slightly deforming the original, making the intersections transversal.

27.1.5. The Alexandrov-Fenchel inequality can be deduced from the Hodge inequality and the theorem on the number of roots. Let us indicate how this is done.

Consider a non-compact algebraic surface defined in $\mathbb{C}^n$ by a general system of $n-2$ polynomial equations $P_3 = 0, \ldots, P_n = 0$ with Newton polyhedra $\Delta_3, \ldots, \Delta_n$. On the surface consider two curves A_1 and A_2 defined respectively by the general polynomial equations $P_1 = 0$ and $P_2 = 0$ with Newton polyhedra Δ_1 and Δ_2. The number of intersection points of the curves A_1 and A_2 on our non-compact surface is equal to the number of solutions of the system of equations $P_1 = P_2 = P_3 = \cdots = P_n = 0$. This number, according to the theorem on a number of roots, is equal to $n!V(\Delta_1, \ldots, \Delta_n)$. Together with the curve A_1 consider a slightly deformed copy of this curve A_1': the curve A_1' is defined by the equation $P_1' = 0$ which contains the same terms as the equation $P_1 = 0$ but with slightly changed coefficient (in particular, the polynomials P_1 and P_1' have the same Newton polyhedron). According to the same theorem, the number of intersection points of the curve A_1 with the curve A_1' equals $n!V(\Delta_1, \Delta_1, \Delta_3, \ldots, \Delta_n)$. Similarly we can construct the curve A_2' for which the number of intersection points with the curve A_2 equals $n!V(\Delta_2, \Delta_2, \Delta_3, \ldots, \Delta_n)$.

The next step consists in the compactification of the non-compact surface $P_3 = P_4 = \cdots = P_n = 0$.

There exists a special compactification of this surface for which the closures $\Gamma_1, \Gamma_1', \Gamma_2, \Gamma_2'$ of the non-compact curves A_1, A_1', A_2, A_2' have no "points at infinity" as intersection points. The absence of "points at infinity", where the curves $\Gamma_1 = \bar{A}_1$ and $\Gamma_1' = \bar{A}_1'$ intersect, means that $\Gamma_1 \cap \Gamma_1' = A_1 \cap A_1'$; a similar relation holds for the other pairs of curves. Therefore, the intersection index and the self-intersection index of the curves Γ_1 and Γ_2 are determined by the formulas

$$\langle \Gamma_1, \Gamma_2 \rangle = n!V(\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n)$$

$$\langle \Gamma_1, \Gamma_1 \rangle = n!V(\Delta_1, \Delta_1, \Delta_3, \ldots, \Delta_n)$$

$$\langle \Gamma_2, \Gamma_2 \rangle = n!V(\Delta_2, \Delta_2, \Delta_3, \ldots, \Delta_n)$$

Substituting these formulas into the Hodge inequality, we obtain the Alexandrov-Fenchel inequality for the polyhedra $\Delta_1, \Delta_2, \ldots, \Delta_n$:

$$V^2(\Delta_1, \Delta_2, \ldots, \Delta_n) \geqslant V(\Delta_1, \Delta_1, \Delta_3, \ldots, \Delta_n) \cdot V(\Delta_2, \Delta_2, \Delta_3, \ldots, \Delta_n)$$

27.2. Hyperbolic Quadratic Forms. The Alexandrov-Fenchel inequality resembles in form the Cauchy-Buniakovski inequality except that the signs of these inequalities are opposite. In this subsection we consider hyperbolic quadratic forms, for which the reciprocals of the Cauchy-Buniakovski inequalities hold.

27.2.1. A quadratic form is said to be *hyperbolic*, if there exist vectors on which it assumes positive values but there exists no two-dimensional plane on which it is positive defined.

***27.2.2. Example.** The quadratic form $x_1^2 + \cdots + x_l^2 - x_{l+1}^2 - \cdots - x_{l+m}^2$, defined in real space of dimension $n \geqslant l + m$ with coordinates $x_1, \ldots, x_n$, is hyperbolic if and only if $l = 1$.

27.2.3. Proposition. *Suppose B is a symmetric bilinear form for which the quadratic form $Q(a) = B(a, a)$ is hyperbolic. Then for any positive vector x (i.e. for any vector x satisfying $B(x, x) > 0$ and any vector y we have the reciprocal Cauchy-Buniakovski inequality*

$$B^2(x, y) \geqslant B(x, x)B(y, y). \tag{1}$$

Proof. Consider the polynomial φ of the second degree in the real variable t defined by the relation $\varphi(t) = Q(tx + y)$. The leading coefficient $B(x, x)$ is positive while the discriminant Δ equals $4[B^2(x, y) - B(x, x)B(y, y)]$. If $\Delta < 0$, then the polynomial φ has no real roots and the form Q is positive definite on the plane generated by the vectors x and y (for $\Delta \neq 0$ the vectors x and y are automatically independent). By hypothesis, no such plane can exist. Therefore $\Delta \geqslant 0$, which was to be proved.

Let us show that inequalities of type (1) are not only necessary but also sufficient for a form to be hyperbolic. Suppose K is a certain cone generated by the linear space L (this means that any element $z \in L$ can be represented in the form $z = x - y$, where x and y are contained in K).

27.2.4. Proposition. *Suppose there exists an interior point x of the cone K satisfying $B(x, x) > 0$ and for any vector $y \in K$ we have the inequality $B^2(x, y) \geqslant B(x, x) \cdot B(y, y)$. Then the quadratic form $Q(a) = B(a, a)$ is hyperbolic.*

Proof. Consider the hyperplane M, B-orthogonal to the vector x (the inclusion $z \in M$ is equivalent to the relation $B(x, z) = 0$). Let us show that the restriction of the quadratic form Q to the hyperplane M is non-positive. Indeed, suppose there is a vector u in M such that the form Q is positive. Then the form Q is positive definite on the plane generated by the vectors x and u. For any vector y (from this plane) non-collinear to the vector x, we have the Cauchy-Buniakovski inequality $B^2(x, y) < B(x, x)B(y, y)$. Such a vector y may be chosen in the cone K, since x is an interior point of the cone. We obtain a contradiction with the hypothesis of the proposition, thus showing that the form Q is non-positive on the hyperplane M. Any two-dimensional plane intersects the hyperplane M. Therefore there exists no plane on which the form Q would be positive definite. Proposition 27.2.4 is proved.

27.2.5. Let us return to the Alexandrov-Fenchel inequality. Convex bodies do not constitute a linear space. It is impossible to subtract them. The formal differences of convex bodies already constitute a linear space, while the set of convex bodies are a cone in this space.

The mixed volume is defined for a cone of convex bodies. By linearity it can be extended on the entire linear space.

Propositions 27.2.3 and 27.2.4 shows that the Alexandrov-Fenchel inequality is equivalent to the bilinear form defined by the formula $B(a, b) =$

$V(a, b, A_3, \ldots, A_n)$, where $A_3, \ldots, A_n \in K$ are fixed convex bodies, being hyperbolic. Hyperbolic forms appear in the theory of algebraic surfaces: the Hodge theorem states that the intersection form of curves on an algebraic surface is hyperbolic. This fact, together with the theorem on the number of roots, constitutes the basis of the algebraic proof of the Alexandrov-Fenchel inequality.

27.3. Remarks on the Theorem Concerning the Number of Roots.

27.3.1. A few words on the disposition of the topics which follow. After some historical remarks (in this subsection) and the introduction of a series of notions in subsection 27.4, the subsections 27.5–27.9 will give the proof of the theorem on the number of roots. The outline of this proof is the following. The system $P_1 = \cdots = P_n = 0$ is viewed as the intersection of the curve $P_1 = \cdots = P_{n-1} = 0$ with the hyperplane $P_n = 0$.

The theorem on the intersection of curves and hypersurfaces (see 27.5.5) reduces (see 27.8) the problem of finding the number of their intersection points to the problem of finding the degrees of the asymptotic lines of the curves and to the determination of their number. In 27.7 we state the curve theorem, which enables us to find the degrees of asymptotic lines of the curve and determine their number. The necessary information on the compactification of curves is given in 27.6. The proof of the curve theorem and the theorem on the number of roots is carried out simultaneously (the sequence of required steps is described in 27.7.3). The theorem on the number of roots in n-dimensional space reduces to the curve theorem in n-dimensional space. The theorem on the curve in n-dimensional space reduces the theorem on the number of roots in $(n-1)$-dimensional space. Here part of the asymptotic lines can be found directly (see 27.9). In order to find the other asymptotic lines, it is necessary to carry out an monomial transformation (see 27.9.4). In subsection 27.10 we prove the necessary statements on general position properties.

***27.3.2. Historical remarks.** Ferdinand Minding (1806–1885), Professor at Derpt University, was the first to apply Newton polyhedra to the problem of determining the number of solutions of a system of two equations in two unknowns. Here we give an exposition of Minding's method.

Suppose $f(x, y) = 0$, $g(x, y) = 0$ is a system of two polynomial equations in two unknowns. Minding excludes the unknown y from the system. In order to do this, he considers a multivalued function $y(x)$, determined by the equation $f(x, y) = 0$ and substitutes it into the second equation. The number of branches of the algebraic function $y(x)$ is equal to the degree of the polynomial f with respect to the variable y. Suppose this degree equals k and $y_1(x), \ldots, y_k(x)$ are the different branches of the function $y(x)$. Multiplying all k branches $g(x, y_i(x))$ of the multivalued function $g(x, y(x))$, we obtain the function $p(x) = \prod_{1 \leqslant i \leqslant k} g(x, y_i(x))$, which is already a single-valued function. The zeros of the function p correspond to the roots of the given system. The algebraic function p is single-valued and therefore rational. Moreover, if the curve $f(x, y) = 0$ has no vertical asymptotes, then the function $p(x)$ for finite x never becomes infinite and therefore is a polynomial. In order to determine the number of roots of the system it is necessary to find the number of zeros of the polynomial p which equals its degree. To determine the degree of the polynomial p, Minding proposes the following method. Compute the leading term of the expansion of the branches $y_i(x)$ of the function $y(x)$ into a series of fractional powers of x (into the so-called Puiseux series) as $x \to \infty$. Then, having substituted the leading terms of the expansion of $y_i(x)$ into $g(x, y)$ compute the leading terms of the expansion of the branches of $g(x, y_i(x))$ and determine their (fractional) degrees. The required degree of the polynomial p is equal to the sum of the degrees of the branches of $g(x, y_i(x))$. This sum is always an integer.

Newton polyhedra have long been used in order to find the leading terms of the expansion of algebraic functions into Puiseux series. Minding uses the Newton polygon of the polynomial $f(x, y)$ in order to find the leading term of the expansion of the branches $y_i(x)$. He notices that if the coefficient of the polynomials f and g are not related by any special relation, then in the substitution into $g(x, y)$ of the branch $y_i(x)$ the leading terms do not cancel. Here the degree of the branch $g(x, y_i(x))$ depends only on the degree of the branch $y_i(x)$ and on the Newton polygon of the polynomial $g(x, y)$. In this case Minding finds the formula for determining the number of roots of the system in terms of the Newton polygons of the polynomial f and g. The geometric meaning of this formula was unknown and became clear only in connection with the appearance of the theorem on the number of roots [BerD].

Minding's work was published in 1841 in Crelle's journal [Min]. The discussion which followed showed that Minding clearly understood the necessity of general position for his formulas to be valid. Magnus gave an example contradicting the Minding formula, see [Mag]. In his answer Minding showed that in the Magnus example the non-degeneracy conditions did not hold.

Our method of proof of the theorem on the number of roots for $n = 2$ is a variation of Minding's method.

***27.3.3.** Minding's theorem attracted the attention of such mathematicians as Liouville and Hermite, but then was forgotten. The next step was carried out a hundred and thirty years later by A. G. Kushnirenko. In 1975 he proved that the number of solutions of a non-degenerate system of n equations in n unknowns possessing identical Newton polyhedra is equal to the volume of this polyhedron multiplied by $n!$ [Kou 1, 2]. A. G. Kushnirenko's proof uses the techniques of commutative algebra and is rather complicated. In the same year of 1975 D. N. Bernstein published the theorem on the number of roots [BerD]. His proof is close to Minding's method. The outline of this proof is the following. It is proposed to introduce the parameter t into the system $f_1 = \cdots = f_n = 0$ under consideration and pass to the system $f_n = t, f_1 = \cdots = f_{n-1} = 0$ (the roots $z^i(t)$ of this system depend on the parameter t). When the parameter t changes from zero to infinity all the roots $z^i(t)$ tend to disappear from the space $(\mathbb{C} - 0)^n$ but their number in the process of motion does not change. Bernstein proposes to determine the number of roots of $z^i(t)$ for very large t by expanding them into a Puiseux series with respect to the small parameter $u = t^{-1}$. First he finds the number of roots for which the Puiseux series are of the form $z_1^i = a_1 u + \cdots$, $z_2^i = a_2 + \cdots, \ldots, z_n^i = a_n + \cdots$, i.e. for which the leading terms of the components are respectively equal to $1, 0, \ldots, 0$. It then turns out that the coefficients $a_2, \ldots, a_n$ satisfy a system in a smaller number of unknowns, whose number of roots is known by induction. The determination of the number of roots of $z^i(u)$ with a different asymptotic with respect to u, $z_1^i = a_1 u^{N_1} + \cdots, \ldots, z_n^i = a_n u^{N_n} + \cdots$ reduces to the previous case when $N_1 = 1$, $N_2 = \cdots = N_n$ by means of an exponential transformation (see 27.9.7).

***27.3.4.** Our proof of the theorem on the number of roots resembles the proof outlined above. To make this similarity even more obvious, let us note that the roots of the system $f_n = t$, $f_1 = \cdots = f_{n-1} = 0$, when t changes, move along the curve $f_2 = \cdots = f_{n-1} = 0$ and that the problem of determining the leading terms of the Puiseux series of the roots $z^i(t)$ is equivalent to the problem of finding the asymptotic lines of the curve $f_1 = \cdots = f_{n-1} = 0$. But our proof is much more detailed. The theorem on the intersection of curves and hypersurfaces and the curve theorem do not appear in [BerD] and are published here for the first time. Moreover, we do not use Puiseux series. They are replaced by the compactification of the algebraic curve. This is simpler, although the difference is not one in principle: the Riemann surface is constituted in fact by all possible Puiseux series.

***27.3.5.** At the present time there are many proofs of the theorem on the number of roots. The toric compactification of the space $(\mathbb{C} - 0)^n$ (see 27.12) makes it possible to apply the arsenal of algebraic geometry to this problem. The theorem on the number of roots is one of the numerous consequences of the computation of toric manifold cohomology [Kh 2]. But the proof given in 27.5–27.9 is one of the most elementary ones.

27.4. Monomials, Monomial Curves, Laurent Polynomials and Their Newton Polyhedra. In this subsection we show that to the product of polynomials corresponds the Minkowski sum of their Newton polyhedra.

27.4.1. The monomials and the Laurent polynomials $f(z)$ in n variables $z = (z_1, \ldots, z_n)$ are always defined and are holomorphic functions in the space $(\mathbb{C} - 0)^n$ consisting of all ordered sequences of n non-zero complex numbers. In the sequel many objects which will interest us will be contained in this space. The support of the Laurent polynomial $f(z) = \sum c_m z^m$, $z^m = z_1^{m_1} \ldots z_n^{m_n}$ is the set $\operatorname{supp} f \subset \mathbb{R}^n$ consisting of those integer vectors $m \in \mathbb{R}^n$ for which $c_m \neq 0$. The Newton polyhedron of the Laurent polynomial f (which we shall denote by $\Delta(f)$) is the convex envelope of the support of f.

***27.4.2. Examples.** 1) The Newton polygon of the equation of the curve $y^2 + a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$ is the triangle with vertices $(0, 2)$, $(0, 0)$ and $(3, 0)$ (we assume that $a_0 \neq 0$ and $a_3 \neq 0$).

2) The Newton polygon of the polynomial $\sum c_m z^m$ (where all the c_m are non-zero) in n variables of degree k is a simplex homothetic to the standard one and determined by the inequalities $m_1 \geqslant 0, \ldots, m_n \geqslant 0$, $\sum m_i \leqslant k$.

3) The Newton polygon of a typical polynomial $\sum c_m z^m$ in n variables, whose

degree with respect to the i-th variable is k_i, is the cuboid defined by the inequalities $0 \leqslant m_i \leqslant k_i$.

27.4.3. The curve in space $(\mathbb{C} - 0)^n$ parametrized by a complex parameter $t \neq 0$ and given by the formulas $z_1 = a_1 t^{N_1}, \ldots, z_n = a_n t^{N_n}$ where $N = (N_1, \ldots, N_n)$ is a vector with integer coordinates and $a = (a_1, \ldots, a_n)$ is a point from $(\mathbb{C} - 0)^n$ is called a *monomial curve* (or line) and is briefly written as $z = at^N$. The vector N is said to be the *degree* of this curve. The coefficient a is not uniquely determined. We do not distinguish the lines $z = at^N$ and $z = a(ct)^N$, $c \in (\mathbb{C} - 0)^n$ which differ only in the choice of parameter. If a vector of degree N is zero, then the monomial curve consists of the point a only (which is uniquely determined).

If we restrict a Laurent polynomial to the polynomial line, we obtain a Laurent polynomial in one complex variable t. For example, if we restrict the monomial cz^m to the line at^N, we obtain the monomial $ca^m t^{\langle N, m\rangle}$. Let us try to specify the leading term of the restriction of the Laurent polynomial $f = \sum c_m z^m$ with Newton polyhedron Δ to the polynomial line $z = at^N$.

27.4.4. By definition, the *support face* of the polyhedron Δ in the direction N is the face Δ_N on which the linear function $\langle N, x\rangle$ assumes its maximum for $x \in \Delta$. This maximum $H_\Delta(N) = \max_{x \in \Delta} \langle N, x\rangle$ is called the *height* of the polyhedron Δ in the direction N. The dependence of the height H_Δ on the vector N is called the *support function* of the polyhedron Δ. The supporting face and the height of a polyhedron are defined for any real (and not only for integer-valued) vectors $N \in \mathbb{R}^n$.

By definition, the *truncation* of the Laurent polynomial $f(z) = \sum_{m \in \Delta} c_m z^m$ with respect to the vector N is the function $f_N(z) = \sum_{m \in \Delta_N} c_m z^m$ (we have omitted the sum of monomials which are not contained in the supporting face Δ_N).

27.4.5. Example. The truncation of the polynomial with respect to the vector $N = (1, \ldots, 1)$ is the homogeneous term of the highest order of this polynomial. The truncation of the Laurent polynomial with respect to the vector $N = 0$ coincides with the original Laurent polynomial $f_0 = f$.

27.4.6. The restriction of the Laurent polynomial $f(z) = \sum c_m z^m$ to the monomial curve at^N is equal to $f_N(a)t^{H_\Delta(N)}$ plus terms of smaller degree in t. Thus the truncation of a polynomial with respect to the vector N determines the leading term in its restriction to a family depending on the parameter a of lines of degree N (for some exceptional values of the parameter a the coefficient $f_N(a)$ vanishes and then the leading coefficient will be a term of smaller degree).

Suppose f and g are arbitrary Laurent polynomials and $h = f \cdot g$.

27.4.7. Proposition. *When Laurent polynomials are multiplied,*

a) *the trunkations (with respect to any integer vector N) are multiplied:* $h_N = f_N \cdot g_N$;

b) *the support functions of the Newton polyhedra are added* $H_{\Delta(h)} = H_{\Delta(f)} + H_{\Delta(g)}$;

c) *the Newton polyhedra are added* $\Delta(h) = \Delta(f) + \Delta(g)$.

Proof. Restrict the polynomials f, g, h to the family of lines $z = at^N$ of fixed degree N and parameter a. Choosing the term of highest degree with respect to t, we obtain the following relation for the product

$$h_N(a)t^{H_{\Delta(h)}(N)} = f_N(a)t^{H_{\Delta(f)}(N)} \cdot g_N(a)t^{H_{\Delta(g)}(N)}$$

(When we claim that the right-hand side contains the leading term, we use the fact that the product of two non-zero polynomials f_N and g_N is a non-zero polynomial). This relation implies statement a) as well as the statement b) for integer arguments of the support functions.

The support functions are positive homogeneous and continuous, therefore statement b) can be generalized to rational and then to arbitrary support functions with vector variables.

Using the additivity of the support functions and statement b), we see that the support functions of the polyhedra $\Delta(h)$ and $\Delta(f) + \Delta(g)$ coincide. Therefore the polyhedra themselves coincide.

27.5. Intersection of Curves and Hypersurfaces. In this subsection we compute the number of intersection points of a curve and hypersurface under the assumption that they have no "points at infinity" where they intersect. The answer involves the supporting function of the Newton polyhedron of the hypersurface's equation.

27.5.1. Some definitions. Support Γ is a compact complex curve (a compact analytic manifold of complex dimension one) and $z_1, \ldots, z_n$ are meromorphic functions on it. These functions determine a holomorphic mapping (defined outside the poles and zeroes of the functions z_i) of the curve $(\mathbb{C} - 0)^n$. A *divisor* of the vector-function $z = (z_1, \ldots, z_n)$ is by definition the (finite) set of points on the curve Γ in which at least one of the functions z_i vanishes or becomes infinite. Suppose d is one of the points of the divisor and u a local parameter on the curve Γ near the point d, such that $u(d) = 0$. Expand the functions into a Laurent series near the point d: $z_i = a_i u^{N_i} + \cdots$, $a_i \neq 0$.

27.5.2. Definition. The *asymptotic line of the image of the curve* $z(\Gamma)$ *related to the point* d of the divisor is the monomial curve $z(t) = at^{-N}$ where $a = (a_1, \ldots, a_n)$, $N = (N_1, \ldots, N_n)$ are the coefficients and the powers of the first terms of the Laurent series of the functions $z_1, \ldots, z_n$.

The choice of sign for the vector N is a matter of convenience. The monomial curve $z(t)$ for large t is close to the image $z(u)$ of the curve Γ for small $u = t^{-1}$. The choice of local parameter near the point d on the curve Γ influences the equation of the asymptotic line. However, lines corresponding to different local parameters on the curve Γ differ only in their parametrization and we shall not distinguish them.

Suppose $f = 0$ is a hypersurface in $(\mathbb{C} - 0)^n$ defined by the Laurent polynomial f with Newton polyhedron Δ. The restriction of the polynomial f to the monomial curve $z(t) = at^N$ possesses an ordinary degree equal to the height $H_\Delta(N)$ of the polyhedron Δ in the direction N. For certain values of the parameter

a the rate of growth of the restriction of the polynomial f to the line $z(t)$ as $t \to \infty$ may fall. For these values of the parameter the monomial line at^N approaches the hypersurface $f = 0$ (the convergence is extremely rapid).

27.5.3. Definition. An *asymptotic line of the hypersurface* $f = 0$ is a monomial curve $z(t) = at^N$, $N \neq 0$ such that the degree of the restriction of the polynomial f to it is less than $H_\Delta(N)$. In other words, $z(t) = at^N$ is an asymptotic line if and only if $N \neq 0$ and $f_N(z(t)) = 0$.

Note that the choice of the polynomial f determining the hypersurface (the polynomial f is defined up to multiplication by a monomial) does not influence the definition of an asymptotic line.

***27.5.4. Remark.** In the definition of an asymptotic line of a hypersurface we can omit the assumption $N \neq 0$. Then the line $z(t) = at^0 = a$ will be asymptotic if $f(a) = 0$. Hence asymptotic lines with degree $N = 0$ may be identified with points on the hypersurface. But asymptotic lines with degree $N \neq 0$ may be viewed as "points at infinity" of the hypersurface $f = 0$ which do not fit in $(\mathbb{C} - 0)^n$. A similar situation may be observed with the asymptotic lines of the image of the curve. They may be related not only to points of the divisor, but to arbitrary points of the curve Γ as well. Then the monomial curve is related to a regular point $\gamma \in \Gamma$ and has degree $N = 0$; it coincides identically with the point $z(\gamma)$ of the image of the curve Γ. Non-trivial asymptotic lines are related only to points of the divisor. They play the role of the images of the point of the divisors which "do not fit" into $(\mathbb{C} - 0)^n$.

27.5.5. Theorem (on the intersection of curves and hypersurfaces). *Suppose none of the asymptotic lines of the image $z(\Gamma)$ of the compact curve Γ are asymptotic lines of the hypersurface $f = 0$. Then the number of points (multiplicity being taken into consideration) where the hypersurface $f = 0$ intersects the image of the curve $z(\Gamma)$ in $(\mathbb{C} - 0)^n$ equals $\sum_{d \in D} H_\Delta(N(d))$, where D is the divisor of the vector function z; $N(d)$ is the degree of the asymptotic line corresponding to the point d of the divisor; Δ is the Newton polyhedron of the Laurent polynomial f and H_Δ is its support function.*

Proof. The intersection points of the curve and hypersurface in $(\mathbb{C} - 0)^n$ correspond uniquely to the zeroes of the compound function $f(z)$ on the curve Γ outside of its divisor D. The multiplicity of intersection points equals the order of the corresponding zero of the function $f(z)$ (recall that the order of a meromorphic function at any point is the minimal degree of the monomial in the Laurent series expansion of the function at this point). On a compact curve, the sum of orders of zeros and poles of any meromorphic function is equal to zero [Spr]. Since the function $f(z)$ has no poles on $\Gamma - D$, the number of zeros of the function $f(z)$ on $\Gamma - D$ (multiplicity taken into consideration) is equal to the sum of the orders taken with the minus sign of the function $f(z)$ at points of the divisor. Minus the order of the function $f(z)$ at the point d of the divisor equals (plus) the degree of the leading term of the restriction of the polynomial f to the asymptotic line $z(t)$ corresponding to the point d. By assumption of the theorem, the asymptotic line $z(t)$ of the curve Γ is not an asymptotic line of the hypersurface $f = 0$. Therefore the degree of the restriction of the polynomial f to the line $z(t)$ equals the height of the Newton polyhedron of a polynomial f along the degree vector of the line $z(t)$.

27.6. Riemann Surfaces (Compactification of Algebraic Curves).

27.6.1. Consider the set defined in $\mathbb{C}^n$ by the compatible system

$$f_1 = \cdots = f_{n-1} = 0, \qquad P \neq 0 \tag{2}$$

of $n - 1$ polynomial equations and one polynomial inequality. We assume that the differentials of the polynomials $f_1, \ldots, f_{n-1}$ are linearly independent at the roots of the system (2). Denote by Π the hypersurface determined by the equation $P = 0$. According to the implicit function theorem, the compatible system (2) defines in $\mathbb{C}^n - \Pi$ an analytic curve, i.e. a one-dimensional complex analytic manifold. This manifold is not compact, however it possesses a compactification. The compactification of an algebraic curve is called a Riemann surface in complex analysis and a normalization of the curve in algebra. We will now define the compactification and give a list of some of its properties without proof.

27.6.2. Definition. The triple Γ, K, z consisting of a compact curve Γ, a finite set $K \subset \Gamma$ and a meromorphic vector-function $z = (z_1, \ldots, z_n)$ of the curve Γ determines a *compactification* of the curve (2) if the map $z\colon \Gamma - K \to \mathbb{C}^n$ is defined on $\Gamma - K$ and establishes a bijective and bianalytic correspondence between $\Gamma - K$ and the curve (2).

27.6.3. Properties. 1) Any curve (2) possesses a compactification.

2) The compactification is unique. This means that if the triples Γ, K, z and $\tilde{\Gamma}$, $\tilde{K}$, $\tilde{z}$ give a compactification of the curve (2), then there exists a bijective and bianalytic correspondence between the curve Γ and the curve $\tilde{\Gamma}$ sending the set K into the set $\tilde{K}$ and the vector function z into the vector-function $\tilde{z}$.

3) None of the points added to the curve (2) under the compactification "fits" in $\mathbb{C}^n - \Pi$. This means that at every point $k \in K$ either one of the components of the vector function z becomes infinite, or $z(k) \in \Pi$.

Thus to every non-compact curve (2) we associate a unique compact curve Γ. The latter can be conveniently imagined as the curve (2) to which we have glued on certain points which "do not fit" into $\mathbb{C}^n - \Pi$. The components of the vector function z are conveniently represented as coordinate functions on the curve (2). In the sequel, we will have this representation in mind. Allowing ourselves a bit of carelessness in our expression, we shall talk about the asymptotic lines of the curve (2), having in mind the asymptotic lines of the image $z(\Gamma)$ of its compactification Γ.

27.6.4. We will mostly have to deal with curves defined in $(\mathbb{C} - 0)^n$ by the system

$$f_1 = \cdots = f_{n-1} = 0, \tag{3}$$

in which f_i are Laurent polynomials (it is assumed that the system is compatible and the differentials of the functions f_i are independent at the zeros of the system).

The space $(\mathbb{C}-0)^n$ is the space $\mathbb{C}^n$ from which the hypersurface Π with equation $z_1 \cdot \ldots \cdot z_n = 0$ has been excluded. Although the functions f_i in (3) are not polynomials, the system (3) is equivalent to a polynomial system in $(\mathbb{C}-0)^n$: the Laurent polynomials, after multiplication by appropriate monomials, become ordinary polynomials.

27.6.5. Lemma. *Suppose Γ, K, z is a triple determining the compactification of the curve* (3). *Then the set K coincides with the divisor of the vector function z.*

Indeed, according to property 3, one of the components of the vector function z vanishes or becomes infinite at the points of the set K.

27.6.6. Lemma. *If the curve $z(t) = at^N$, $N \neq 0$, is an asymptotic line of the curve* (3), *then the truncation of the system* (3) *according to the vector N is "annihilated" by this curve, i.e. $f_{1N}(z(t)) \equiv \cdots \equiv f_{n-1,N}(z(t)) \equiv 0$.*

Proof. The image $z(\Gamma)$ of the compactification of the curve (2) satisfies the system (3). Developing the zero functions $f_i(z)$ into Laurent series with respect to the small parameter u in a neighbourhood of the point d of the divisor and setting those terms which are the leading terms with respect to the large parameter $t = u^{-1}$ equal to zero, we obtain the required identity.

Lemma 27.6.6 imposes strong restrictions on the degree vector of the asymptotic line. This is because for a vector in general position the truncations f_{iN} of the functions f_i are monomials (the maximum of the scalar products with vector N in general position is obtained on the vertices of the Newton polyhedra of the functions f_i) and the system of truncations $f_{1N} = \cdots = f_{n-1,N} = 0$ is incompatible in $(\mathbb{C}-0)^n$. Historically it is precisely restrictions of this sort which led to the definition of the polyhedron (be more precise, the polygon) due to Newton.

27.7. Statements of the Theorems and Their Sequence of Proof. In this subsection we state a series of theorems among which the main role is played by the theorem on the number of roots.

We begin with different variants of the definition of non-degenerate systems of equations.

27.7.1. Definition. The system of equations $f_1 = \cdots = f_k = 0$ in $(\mathbb{C}-0)^n$ is called *non-degenerate* if at every zero of this system the differentials of the functions $f_1, \ldots, f_k$ are independent.

27.7.2. Definition. The system of equations $f_1 = \cdots = f_k = 0$ in $(\mathbb{C}-0)^n$ is called *Δ-non-degenerate* if for any vector N with integer coordinates the truncated system $f_{1N} = \cdots = f_{kN} = 0$ is non-degenerate in $(\mathbb{C}-0)^n$.

Note that any Δ-non-degenerate system is non-degenerate: for the zero vector $N = 0$ the truncated system coincides with the original one.

27.7.3. Definition. The system of equations $f_1 = \cdots = f_k = 0$ in $(\mathbb{C}-0)^n$ is said to be *strongly Δ-non-degenerate* if any of its subsystems $f_{i_1} = \cdots = f_{i_r} = 0$ is Δ-non-degenerate (here $i_1, \ldots, i_r$ is any increasing sequence of indices $1 \leq i_1 < \cdots < i_r \leq h$).

27.7.4. Definition. A system of n equations in n unknowns $f_1 = \cdots = f_n = 0$ is called *Δ-non-degenerate at infinity*, if for any vector $N \neq 0$ with integer coordinates the truncated system is incompatible.

From the formal point of view, the Δ-non-degeneracy requires an infinite number of conditions: it is required that for any vector N the corresponding truncated system be non-degenerate. In fact only for a finite number of vectors N do we obtain different truncated systems and the number of conditions of Δ-non-degeneracy is actually finite. A similar remark applies to strong Δ-non-degeneracy and to Δ-non-degeneracy at infinity.

27.7.5. Theorem. *For almost all possible families of coefficients of Laurent polynomials $f_1, \ldots, f_k$ with fixed Newton polyhedra, the system of equations $f_1 = \cdots = f_k = 0$ is strongly non-degenerate on* $(\mathbb{C} - 0)^n$.

It follows from Theorem 27.7.5 that all systems of equations with given Newton polyhedra are Δ-non-degenerate and (if the number of equations equals the number of unknowns) are Δ-non-degenerate at infinity, since strong Δ-non-degeneracy of systems automatically implies their Δ-non-degeneracy and their Δ-non-degeneracy at infinity.

27.7.6. Theorem (on the number of roots). *The number of roots in* $(\mathbb{C} - 0)^n$ *of a strongly Δ-non-degenerate system of n equations in n unknowns $f_1 = \cdots = f_n = 0$ is equal to the mixed volume of the Newton polyhedra of the Laurent polynomials $f_1, \ldots, f_n$ multiplied by n!*

27.7.7. Remark. The theorem on the number of roots is valid also for Δ-non-degenerate systems. It remains valid even for Δ-non-degenerate systems at infinity if only the roots of the system of equations are counted according to their multiplicity (Δ-non-degeneracy guarantees that all the roots are simple roots, Δ-non-degeneracy at infinity guarantees that the roots of the system do not "leave" $(\mathbb{C} - 0)^n$, however they may be multiple). Let us dwell on certain corollaries of the theorem on the number of roots.

27.7.8. Corollary. The number of roots in $(\mathbb{C} - 0)^n$ of a strongly Δ-non-degenerate system of equations with the same Newton polyhedra is equal to the volume of their Newton polyhedra multiplied by n!

It is precisely this corollary that Kushnirenko proved before the theorem on the number of roots had been established (see 27.3).

27.7.9. Corollary. The number of roots in the entire space $\mathbb{C}^n$ of a strongly Δ-non-degenerate system of polynomial equations with fixed Newton polyhedra is equal to the mixed volume of the Newton polyhedra multiplied by n! if we assume in addition that all the polynomials have non-zero constant terms.

In order to deduce corollary 27.7.9 of the theorem on the number of roots it suffices to check that under the assumptions of corollary 27.7.9 the system of equation has no roots in the coordinate plane. We will not stop to make this verification. Note that in corollary 27.7.9 the condition on the existence of non-zero constant terms cannot be omitted (although it may be weakened somewhat). For example the equation $x^n = 0$ has a Newton polygon constiting of one point n. In full agreement with the theorem on the number of roots, this equation has no roots in $\mathbb{C} - 0$, although of course in $\mathbb{C}$ it has a solution.

27.7.10. Corollary (the Bézout theorem). The number of roots in $\mathbb{C}^n$ of a general system of polynomial equations $P_1 = \cdots = P_n = 0$ of degrees $k_1, \ldots, k_n$ respectively equals $k_1 \cdot \ldots \cdot k_n$.

For the proof, it suffices to compute the mixed volume of a system of n simplices homothetic to the standard one.

Consider the curve defined in $(\mathbb{C} - 0)^n$ by the strongly non-degenerate system

$$f_1 = \cdots = f_{n-1} = 0 \tag{4}$$

with Newton polyhedra $\Delta_1, \ldots, \Delta_{n-1}$.

Theorem 27.7.11 stated below makes it possible to determine the degrees of all asymptotic lines of the curve (4), find their number and write out systems of equations whose solutions are the asymptotic lines.

It is convenient to introduce the following notation: for any non-zero vector N with integer coordinates denote by $S(N)$ the number equal to the mixed $(n-1)$-dimensional volume of the supporting faces of the polyhedra $\Delta_1, \ldots, \Delta_{n-1}$ in the direction of N multiplied by $(n-1)!$, i.e.

$$S(N) = V_{n-1}(\Delta_{1N}, \ldots, \Delta_{n-1,N}) \cdot (n-1)!$$

27.7.11. Theorem (curve theorem). 1. *The system* (4) *is incompatible if and only if for all non-zero vectors N with integer coordinates the number $S(N)$ vanishes.* 2. *If the system* (4) *is compatible, then the curve that it determines has no asymptotic lines with cancellable vectors degree N (i.e. degree vectors whose components $N_1, \ldots, N_n$ have a common divisor different from ± 1).* 3. *The asymptotic lines of the curve* (4) *with non-cancellable vector of degree N are connectivity components of the set defined in $(\mathbb{C} - 0)^n$ by the system $f_{1N} = \cdots = f_{n-1,N} = 0$; their number is equal to the number $S(N)$ divided by the length of the vector N.*

***27.7.12. Remark.** Theorem 27.7.11 remains valid for Δ-non-degenerate system (4) also.

***27.7.13. Example.** On the plane consider the curve $y^2 + a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$. Its Newton polyhedron is the triangle with vertices (0, 2), (0, 0), (3, 0). The number $S(N)$ differs from zero only for three non-cancellable vectors $N_1 = (-1, 0)$, $N_2 = (0, -1)$, $N_3 = (3, 2)$. The corresponding truncated equations are $f_{N_1} = y^2 + a_0 = 0$, $f_{N_2} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0$, $f_{N_3} = y^2 + a_3 x^3 = 0$. The first of these truncations determines two horizontal asymptotic lines $y = \pm\sqrt{-a_0}$ corresponding to the intersection points of the curve with the y axis. The second truncation $f_{N_2} = 0$ determines three vertical lines corresponding to the intersection points of our curve with the x axis. Finally, the third truncation determines a unique non-trivial asymptotic line $y^2 + a_3x^3 = 0$ with parametric equation $y = \sqrt{-a_3}t^3$, $x = t^2$. This line corresponds to the point on the divisor of the vector function (x, y) on the compactification of the curve, near which we have the expansion

$$x = u^{-2} + \cdots, \qquad y = \sqrt{-a_3}u^{-3} + \cdots.$$

***27.7.14. Remark.** When we know the asymptotic lines of the curve $f(x, y) = 0$, we can express y approximately in terms of x as $x \to \infty$ or $x \to 0$. Thus, in the previous example, solving the equation of the asymptotic line $y^2 + a_3x^3 = 0$, we obtain $y \sim \sqrt{-a_3}\,x^{3/2}$ as $x \to \infty$. Theorem 27.7.11 thus makes it possible to find the main terms of the expansion of y in fractional degrees of x as $x \to \infty$ and $x \to 0$ by using the Newton polygon of the equation $f(x, y) = 0$. It is precisely for the solution of this problem that Newton used the polygons of the equations.

27.7.15. Let us describe the sequence of proofs of the theorems stated above. Theorem 27.7.5 will be proved in subsection 27.10. The theorem 27.7.6 on the number of roots and the curve theorem 27.7.11 will be proved by simultaneous induction in the following order.

Step I-B. The theorem on the number of roots is proved in $(\mathbb{C} - 0)^1$.

Step n-A. The curve theorem in $(\mathbb{C} - 0)^n$ is deduced from the theorem on the number of roots in $(\mathbb{C} - 0)^{n-1}$.

Step n-B. The theorem on the number of roots in $(\mathbb{C} - 0)^n$ is deduced from the curve theorem in $(\mathbb{C} - 0)^n$.

Step I-A is absent because there are no curves in one-dimensional space. The steps n-A and n-B will be carried out in 27.8 and 27.9. Here we carry out the step I-B.

The number of non-zero roots of a Laurent polynomial in one variable $\sum_{\underline{m} \leq m \leq \bar{m}} a_m z^m$ equals $\bar{m} - \underline{m}$. The length of the Newton polyhedron of this polynomial, i.e. of the segment $[\underline{m}, \bar{m}]$ also equals $\bar{m} - \underline{m}$. Step I-B has been carried out.

27.8. Deduction of the Theorem on the Number of Roots from the Curve Theorem.

27.8.1. Consider the strongly Δ-non-degenerate system of equations

$$f_1 = \cdots = f_n = 0 \tag{5}$$

with Newton polyhedra $\Delta_1, \ldots, \Delta_n$. The number of solutions of the system in $(\mathbb{C} - 0)^n$ is equal to the number of intersection points in $(\mathbb{C} - 0)^n$ of the curve

$$f_1 = \cdots = f_{n-1} = 0 \tag{6}$$

with the hypersurface $f_n = 0$. The theorem on the intersection of curves and hypersurfaces (see 27.5.5) reduces the problem of computing this number to the problem of finding the degrees of the asymptotic lines of the curve (6), which in turn is solved by the curve theorem. Let us see this in more detail.

27.8.2. We shall need the well-known inductive formula (see (6) in subsection 25.2.3) for computing the mixed volumes

$$V(\Delta_1,\ldots,\Delta_n) = \frac{1}{n!}\sum_{\|\rho\|=1} S(\rho)H_{\Delta_n}(\rho). \qquad (7)$$

Here, as before, $S(\rho)$ is the mixed $(n-1)$-dimensional volume of the supporting faces of the polyhedra $\Delta_1,\ldots,\Delta_{n-1}$ in the direction ρ multiplied by $(m-1)!$, while $H_{\Delta_n}(\rho)$ is the height of the polyhedron Δ_n in the direction ρ. Although the sum is taken over an infinite number of points of the sphere $\|\rho\| = 1$, there is actually only a finite number of non-zero terms—the terms corresponding to vectors ρ orthogonal to the $(n-1)$-dimensional faces of the polyhedron (the sums $\Delta_1 + \cdots + \Delta_{n-1}$). Note that for a polyhedron with integer coordinate vertices, the sum in formula (7) must be taken only over vectors of the form $\rho = N/\|N\|$, where N is a non-zero vector with integer coordinates (the half line which originates at the point O and is orthogonal to the $(n-1)$-dimensional face of the polyhedron $\Delta_1 + \cdots + \Delta_{n-1}$ with integer vertices contains integer points).

27.8.3. Let us return to the system (5). According to the theorem 27.5.5 on the intersection of curves and hypersurfaces the number of roots of this system equals

$$\sum_{d\in D} H_{\Delta_n}(N(d)),$$

where H_{Δ_n} is the support function of the Newton polyhedron Δ_n of the function f_n, while $N(d)$ is the degree vector of the asymptotic line of the curve (6) corresponding to the point d of the divisor $\mathcal{D}$ on the compactification of the curve (6).

Further we will check that the curve (6) and the hypersurface $f_n = 0$ have no common asymptotic lines and therefore theorem 27.7.11 is applicable.

According to the curve theorem (step n-A), for the non-cancellable vector $N \neq 0$, the number of points in the divisor to which asymptotic lines of degree N correspond equals $S(N)/\|N\|$. Using the homogeneity of the support function H_{Δ_n}, we obtain

$$\sum_{d\in D} H_{\Delta_n}(N(d)) = \sum_{\|\rho\|=1} S(\rho)H_{\Delta_n}(\rho), \qquad (8)$$

where the sum in the right-hand side is taken over the vector ρ of the form $\rho = N/\|N\|$, while N is an uncancellable non-zero vector with integer coordinates.

Comparing formulas (7) and (8), we see that the number of solutions of the system (5) equals $n!\,V(\Delta_1,\ldots,\Delta_n)$.

27.8.4. In the argument just carried out, it was assumed that the system (6) is compatible. If the system (6) is not compatible, then of course the entire system (5) is not either. On the other hand, if the system (6) is not compatible, then, according to the curve theorem, all the numbers $S(N)$ vanish and, using the inductive formula for mixed volume, we see that the number $n!\,V(\Delta_1,\ldots,\Delta_n)$ also vanishes.

27.8.5. Thus to conclude step n-B it remains to check the absence of common asymptotic lines of the curve (6) and the hypersurface $f_n = 0$. The relations $f_{1N}(z(t)) \equiv \cdots \equiv f_{n-1,N}(z(t)) \equiv 0$ hold on the asymptotic line $z(t) = at^N$ (see Lemma 27.6.6). If the line $z(t)$ is an asymptotic one for the hypersurface $f_n = 0$, then among the solutions of the system of n equation in n unknowns $f_{1N} = \cdots = f_{nN} = 0$ we can find the curve $z(t)$. This contradicts the fact that this system is non-degenerate, since a non-degenerate system of n equations in n unknowns can only possess discrete roots. Step n-B has been carried out.

27.9. The Curve Theorem.

27.9.1. In order to begin step n-A, let us find the number of asymptotic lines of degree $(-l, 0, \ldots, 0)$, where l is a natural number, for a degenerate curve. Such asymptotic lines correspond to intersection points of the curve with the plane $z_1 = 0$ in which the other coordinate functions $z_2, \ldots, z_n$ do not vanish. The number of intersection points can be computed by using the theorem on the number of roots in $(n-1)$-dimensional space. Let us begin carrying out this outline.

27.9.2. Denote by ξ the vector $(-1, 0, \ldots, 0)$ and by G the hypersurface in $\mathbb{C}^n$ determined by the relation $z_1 = 0$, $z_2 \neq 0, \ldots, z_n \neq 0$. The hypersurface G is isomorphic to the space $(\mathbb{C}\backslash 0)^{n-1}$ with coordinates $z_2, \ldots, z_n$.

The Laurent polynomial f cannot always be restricted to the hypersurface G. To do this it is necessary that f contain no monomials containing z_1 in negative powers. If the polynomial f contains only monomials with positive powers of z_1, then the restriction of f to G equals zero.

We shall say that f is *arranged* with respect to z_1, if it contains no monomials with negative powers of z_1, but contains monomials possessing of zero degree with respect to z_1. Every Laurent polynomial may be arranged with respect to z_1 by multiplying it by an appropriate power of the variable z_1. This does not change the zero level surface of the polynomial in $(\mathbb{C} - 0)^n$, while the Newton polyhedron is moved parallel to itself.

Denote by g the restriction of the Laurent polynomial arranged with respect to z_1, to the hypersurface G, so that $g(z_2, \ldots, z_n) = f(0, z_2, \ldots, z_n)$. The Newton polyhedron of the polynomial g coincides with the support face Δ_ξ of the Newton polyhedron of the polynomial f along the vector ξ. The truncation f_ξ of the function f along the vector ξ does not depend on the variable z_1 and has the same restriction to the hypersurface G, namely $f_\xi|_{z_1=0} = f|_{z_1=0} = g$.

27.9.2. Lemma. *Suppose the Laurent polynomials $f_1, \ldots, f_k$, $k \leqslant n$ are arranged with respect to the variable z_1 and the system $f_1 = \cdots = f_k = 0$ is Δ-non-degenerate (strongly Δ-non-degenerate) in $(\mathbb{C} - 0)^n$. Then the restrictions g_i of the functions f_i to the hypersurface G constitute a Δ-non-degenerate (strongly Δ-non-degenerate) system $g_1 = \cdots = g_k = 0$ in $(\mathbb{C} - 0)^{n-1}$ (the space $(\mathbb{C} - 0)^{n-1}$ is realized as the hypersurface G).*

Proof. First let us prove that the system $g_1 = \cdots = g_k = 0$ is non-degenerate in $(\mathbb{C}-0)^{n-1}=G$. Consider the auxiliary system $f_{1\xi}=\cdots=f_{k\xi}$ in $(\mathbb{C}-0)^n$. The auxiliary system does not in fact differ from the original one: the only difference is that the auxiliary system is considered in a space of dimension less by one, but the functions $f_{i\xi}$ do not depend on the extra variable z_1 and the restrictions of these functions to the hyperplane $z_1=0$ coincide with the functions $g_{i\xi}$. Clearly the auxiliary system is non-degenerate at the same time as the original one. But the auxiliary system is non-degenerate by assumption. In a similar way, one proves the non-degeneracy of all the truncations of the system $g_1=\cdots=g_k=0$: in fact they do not differ from the specially chosen truncated systems $f_1=\cdots=f_k=0$ in $(\mathbb{C}-0)^n$, which were non-degenerate by hypothesis. Lemma 27.9.2 is proved.

Denote by Π the hypersurface in $\mathbb{C}^n$ determined by the equation $z_2\cdot\ldots\cdot z_n=0$. Consider the set A defined in $\mathbb{C}^n-\Pi$ by the system

$$f_1=\cdots=f_{n-1}=0. \tag{9}$$

in which f_i is a Laurent polynomial arranged with respect to z_1 with Newton polyhedron Δ_i.

27.9.3. Lemma. *If the system* (9) *is compatible and strongly Δ-non-degenerate, then the set A is an analytic curve in $\mathbb{C}^n-\Pi$. This curve intersects the hypersurface G transversally, where G is determined in $\mathbb{C}^n-\Pi$ by the equation $z_1=0$. The number of intersection points of the curve A and the hypersurface G equals $S(\xi)$.*

Proof. The space $\mathbb{C}^n-\Pi$ is the union of the space $(\mathbb{C}-0)^n$ and the hypersurface G. At the points of the set A, contained in $(\mathbb{C}-0)^n$, the differentials of the functions $f_1,\ldots,f_{n-1}$ are linearly independent (by definition of Δ-non-degeneracy). In a neighbourhood of these points, according to the theorem on implicit functions, the set A is an analytic curve. Now consider the points of the set $A\cap G$. These points satisfy the system $z_1=0, f_1=\cdots=f_{n-1}=0, z_2\neq 0,\ldots, z_n\neq 0$. Substituting $z_1=0$ into the other equations, we obtain the equivalent system $g_1=\cdots=g_{n-1}=0, z_2\neq 0,\ldots,z_n\neq 0$. According to Lemma 27.9.2 this system is strongly Δ-non-degenerate. The number of roots of this system, according to the theorem on the number of roots (obtained in step $(n-1)$-B of the proof) equals $S(\xi)$. At the roots of the system, the restrictions of the differentials $df_1,\ldots,df_{n-1}$ to the plane $z_1=0$ are linearly independent. Indeed, $df_i|_{z_1=0}=dg_i$, while the independence of the differentials dg_i at the roots of the system was proved in Lemma 27.9.2. Therefore, first of all, in a neighbourhood of points of the set $A\cap G$, we can also apply the implicit function theorem, so that the set A is an analytic curve. Secondly, this curve intersects the hypersurface G transversally. Lemma 27.9.3 is proved.

27.9.4. Lemma. *If $S(\xi)\neq 0$, then any strongly Δ-non-degenerate system $f_1=\cdots=f_{n-1}=0$ is compatible in* $(\mathbb{C}-0)^n$.

Indeed, according to the theorem on the number of roots, the set $A\cap G$ is non-empty in this case. However, near every point of the set $A\cap G$ the system is compatible in $(\mathbb{C}-0)^n$ according to the implicit function theorem.

27.9.5. Lemma. *The curve B determined in* $(\mathbb{C}-0)^n$ *by the compatible Δ-non-degenerate system* $f_1 = \cdots = f_{n-1} = 0$ *has no asymptotic lines of degree* l_ξ, *where* $l > 1$ *and* $\xi = (-1,0,\ldots,0)$. *The asymptotic lines of degree* ξ *are straight lines parallel to the first coordinate axis determined by the system* $f_{1\xi} = \cdots = f_{n-1,\xi} = 0$. *Their number is equal to* $S(\xi)$.

Proof. We shall assume that the Laurent polynomials $f_1,\ldots,f_{n-1}$ are arranged with respect to the variable z_1 (in the converse case they must be multiplied by the appropriate powers of the variable z_1). Consider the auxiliary non-compact curve A determined by the system $f_1 = \cdots = f_{n-1} = 0$, $z_2 \neq 0,\ldots, z_n \neq 0$. The curve A contains the original curve A and differs from it by the finite set $A \cap G$. The vector function $z = (z_1,\ldots,z_n)$ has no poles on the non-compact curve A and the coordinate functions $z_1,\ldots,z_n$ do not vanish on it. Therefore the divisor of the vector function z on the curve A consists of those points of the set $A \cap G$ where the function z_1 vanishes. According to Lemma 27.9.3, the number of these points is equal to $S(\xi)$. According to the same lemma, the curve A intersects the hypersurface G transversally, therefore at these points the function z_1 has a first order zero. Therefore, to the points of the set $A \cap G$ correspond asymptotic lines of degree $\xi = (-1,0,\ldots,0)$. Clearly these lines are straight lines into which the set determined in $(\mathbb{C}-0)^n$ by the system $f_{1\xi} = \cdots = f_{n-1,\xi} = 0$ falls apart. Let us show that the asymptotic line of degree $l\xi$ for $l \geqslant 1$ cannot correspond to any "point at infinity" d added to the curve A under compactification. Indeed, in the converse case we would have the expansions $z_1 = a_1 u^l + \cdots$, $z_2 = a_2 + \cdots, \ldots$, $z_n = a_n + \cdots$, in a neighbourhood of the point d. The vector function z is defined at the point d, $z(d) = (0, a_2,\ldots,a_n)$, however $z(d) \notin \Pi$. We have obtained a contradiction with property 3) in 27.6.3. In order to conclude the proof, it remains to notice that the curve A and the original curve B have the same asymptotic lines. This follows from the uniqueness of compactification: the compactification of the curve A is also the compactification of the original curve B.

27.9.6. Now we come to the conclusion of step n-A. To carry it through, let us find the number of asymptotic lines of degree $N = l\eta$ of the curve in $(\mathbb{C}-0)^n$. In this subsection l denotes a natural number, while $\eta \neq 0$ is a non-cancellable vector with integer coordinates. By using a special monomial coordinate transformation, the general case may be reduced to the case of the vector $\eta = \xi = (-1,0,\ldots,0)$ considered in the previous subsection. Since such coordinate transformations play an important role in the theory of Newton polyhedra, we shall dwell on this in more detail.

27.9.7. Consider two copies of the space $(\mathbb{C}-0)^n$: the space $(\mathbb{C}-0)^n_z$ with coordinate functions $z_1,\ldots,z_n$ and the space $(\mathbb{C}-0)^n_w$ with coordinate functions $w_1,\ldots,w_n$.

The map $(\mathbb{C}-0)^n_w \to (\mathbb{C}-0)^n_z$ defined by the formulas

$$z_1 = w_1^{q_{11}} \cdot \ldots \cdot w_n^{q_{1n}} \qquad (10)$$
$$z_z = w_1^{q_{n1}} \cdot \ldots \cdot w_n^{q_{nn}},$$

where $\{q_{ij}\} = Q$ is a unimodular matrix (a matrix with integer entries and determinant 1) is said to be a *monomial transformation* and is briefly denoted by the formula $z = w^Q$. The transformation (10) is inversible: the inverse transformation is of the form $w = z^{Q^{-1}}$.

The monomial transformation $z = w^Q$ sends any monomial curve $w = at^N$ into the monomial curve $z = a^Q t^{QN}$, the degree vector N undergoing a linear transformation with matrix Q. The degrees of the monomials are transformed by the dual transformation: the monomial z^m becomes the monomial w^{Q^*m}, where Q^* is the matrix dual to the matrix Q. The Laurent polynomial $f(z) = \sum c_m z^m$ under monomial transformation $z = w^Q$ becomes the Laurent polynomial $\sum c_m w^{Q^*m}$. The Newton polyhedron Δ is transformed then into the Newton polyhedron $Q^*\Delta$. It is easy to check that under monomial transformations non-degeneracy, Δ-non-degeneracy and strong Δ-non-degeneracy of systems of Laurent equations is preserved.

27.9.8. Lemma. *Suppose the curve A has a unique asymptotic line $z(t) = at^N$ and the curve B is obtained from the curve A by the monomial transformation $z = w^Q$. Then the curve B has a unique asymptotic line $w(t) = a^{Q^{-1}}t^{Q^{-1}N}$.*

The statement of Lemma 27.9.8 is intuitively clear: under the monomial transformation $z = w^Q$ the monomial curve $z(t) = at^N$ becomes the monomial curve $w(t) = a^{Q^{-1}}t^{Q^{-1}N}$. For a formal proof, it is useful to make the following remark: if the triple (Γ, D, z), where Γ is a compact curve, z a vector function on it and D its divisor, determines the compactification of the curve A, then the triple (Γ, E, w), where w is a vector function on Γ such that $z = w^Q$ and E is its divisor, determines the compactification of the curve B and $D = E$.

27.9.9. Lemma. *For any two non-cancellable non-zero vectors with integer coordinates there exists a unimodular matrix sending one of these vectors into the other. In particular, for the given non-cancellable vector η and the vector $\xi = (-1, 0, \ldots, 0)$, there exists a matrix Q such that $\eta = Q\xi$.*

We omit the proof of the arithmetical lemma 27.9.9.

Now suppose A is the set determined in $(\mathbb{C} - 0)^n_z$ by the strongly Δ-non-degenerate system $f_1 = \cdots = f_{n-1} = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_{n-1}$ and η is a fixed non-cancellable vector.

27.9.10. Lemma. *If the number $S(\eta)$ is not zero, then the set A is non-empty. A non-empty set A is an analytic curve. This curve has no asymptotic lines of degree $l\eta$ for $l > 1$. The number of asymptotic lines of the curve A of degree η equals $S(\eta)/\|\eta\|$. These asymptotic lines are the connectivity components of the sets determined in $(\mathbb{C} - 0)^n$ by the system*

$$f_{1\eta} = \cdots = f_{n-1,\eta} = 0. \tag{11}$$

Proof. Consider the *monomial transformation* $z = w^Q$ where Q is a unimodular matrix sending the vector $\xi = (-1, 0, \ldots, 0)$ into the vector η, $\eta = Q\xi$. Under this transformation, the set A is mapped into the set B determined by the system $f_1(w^Q) = \cdots = f_{n-1}(w^Q) = 0$ with Newton polyhedra $\tilde{\Delta}_1 = Q^*\Delta_1, \ldots, \tilde{\Delta}_{n-1} =$

$Q^*\Delta_{n-1}$. The asymptotic lines of the curve A of degree $l\eta$ correspond to the asymptotic lines of the curve B of degree $l\xi$ and we can apply the results of Lemma 27.9.5. To complete the proof of Lemma 27.9.10, it remains only to transform the answer: the number of asymptotic lines of curve A of degree η, according to the results of 27.9, is equal to $\tilde{S}(\xi) = (n-1)!\, V_{n-1}(\tilde{\Delta}_{1\xi}, \ldots, \tilde{\Delta}_{n-1,\xi})$. To transform the answer, we shall require a simple lemma from linear algebra.

Suppose $Q: \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation of Euclidean space $\mathbb{R}^n$ preserving volume (i.e. $\det Q = 1$). Suppose $L \subset \mathbb{R}^n$ is some k-dimensional subspace and M is its image under the map Q. The dual transformation Q^* sends the orthogonal complement $M^\perp$ to the subspace M into the orthogonal complement $L^\perp$ to the subspace L and we have the following

27.9.11. Lemma. *The transformation $Q: L \to M$ increases the k-dimensional volume element a number of times equal to the number of times Q^* diminishes the $(n-k)$-dimensional volume element (in the transformation $Q^*: M^\perp \to L^\perp$).*

Applying Lemma 27.9.11 to our situation, choosing L to be the one-dimensional subspace spanned by the vector ξ, we see that the operator Q sends ξ into η, i.e. increases length on L $\|\eta\|$ times. It follows from Lemma 27.9.11 that $\tilde{S}(\xi) = S(\eta)/\|\eta\|$, as required to transform our answer.

27.9.12. Now, in order to conclude step n-A, it remains to check the compatibility condition of the strongly Δ-non-degenerate system (9). We saw above that if at least one of the numbers $S(N)$ for $N \neq 0$ is not zero, then the implicit function theorem, together with the theorem on the number of roots, proves the compatibility of the system. It remains to check that we have the following

27.9.13. Lemma. *For a compatible system of equations* (11), *there is a vector $\eta \neq 0$ such that $S(\eta) \neq 0$.*

Proof. If all the numbers $S(\eta)$ are zero, then, according to Lemma 27.9.10, the curve (11) has no asymptotic lines at all. This is impossible since the vector function $z = (z_1, \ldots, z_n)$ on a compactification of the curve (11) has a non-zero divisor (any meromorphic function on a compact curve other than a constant always has poles). Lemma 27.9.13 is proved.

27.10. General (Typical) Systems of Algebraic Equations. In this subsection we show that a general system of equations with fixed Newton polyhedra is strongly Δ-non-degenerate (Theorem 27.7.5) and point out certain sharpened versions.

27.10.1. Suppose S is a finite set of points with integer coordinates in $\mathbb{R}^n$ containing s elements. The set of Laurent polynomials with support S may be identified with the s-dimensional space of coefficients in $(\mathbb{C} - 0)^s$. To do this to each Laurent polynomial $f = \sum_m c_m z^m$ we assign the set of its coefficients $\{c_m\}$. We shall say that the property is valid for almost all Laurent polynomials with support S if it holds for polynomials corresponding to the points of the space of coefficients $(\mathbb{C}-0)^s$ not contained in any excluded subset of zero measure. We shall say that a certain property holds for almost all Laurent polynomials with given Newton polyhedron Δ, if it holds for almost all Laurent polynomials with support equal to the set of integer coordinate points of the polyhedron Δ. In a similar way, we define what is meant by a property holding for almost all Laurent polynomials $f_1, \ldots, f_k$ with given Newton polyhedra $\Delta_1, \ldots, \Delta_k$ or, in more general terms, with given supports $S_1, \ldots, S_k$.

27.10.2. Lemma. *Suppose all the supports $S_1, \ldots, S_k$ contain the origin. Then for almost all Laurent polynomials $f_1, \ldots, f_k$ with these supports the system $f_1 = \cdots = f_k = 0$ is non-degenerate.*

Proof. Since the support S_i contains the origin, the Laurent polynomial f_i contains a non-zero constant term $c_i \neq 0$. For any number $a_i \neq c_i$, the Laurent polynomial $f_i - a_i$ has the same support S_i. Further for a fixed system of functions $f_1, \ldots, f_k$ for almost all finite sequences of numbers $a_1, \ldots, a_k$ the system $f_1 = a_1, \ldots, f_k = a_k$ is non-degenerate. This statement is the content of Sard's theorem (a simple proof of this theorem may be found in the book [Mil 3]). Lemma 27.10.2 follows from this statement.

27.10.3. Theorem. *For almost all Laurent polynomials $f_1, \ldots, f_k$ with support $S_1, \ldots, S_k$ the system $f_1 = \cdots = f_k = 0$ is strongly Δ-non-degenerate in $(\mathbb{C}-0)^n$.*

Proof. Strongly Δ-non-degeneracy consists in the non-degeneracy of the finite number of systems of equations which are various truncations of various subsystems of the given system of equation. Each of these systems in non-degenerate almost everywhere. Indeed, if the supports of the Laurent polynomials of such a system contain the origin, this follows from Lemma 27.10.2. If they don't contain the origin, then the Laurent polynomial system may be modified by appropriate monomials so that their supports contain the origin. The multiplication of the Laurent polynomial by a monomial moves its support parallel to itself but does not influence the disposition of the roots of the system of equations in $(\mathbb{C}-0)^n$ and their non-degeneracy. Theorem 27.10.3 is proved.

In the sequel we shall need the more general

27.10.4. Theorem. *Suppose $g_1 = \cdots = g_l = 0$ is a fixed Δ-non-degenerate (strongly Δ-non-degenerate) system of equations in $(\mathbb{C}-0)^n$ and $S_1, \ldots, S_k$ is a fixed family of supports. Then for almost all Laurent polynomials $f_1, \ldots, f_k$ with supports $S_1, \ldots, S_k$ the system $f_1 = \cdots = f_k = g_1 = \cdots = g_l = 0$ is Δ-non-degenerate (strongly Δ-non-degenerate).*

The proof of theorem 27.10.4 is an almost word for word repetition of that of Theorem 27.10.3. It is only necessarily to change the statement of Lemma 27.10.2 (and in its proof to apply Sard's theorem to the system $f_1 = a_1, \ldots, f_k = a_k$ on the manifold $g_1 = \cdots = g_l = 0$ in $(\mathbb{C}-0)^n$).

***27.10.5. Remark.** Theorems 27.10.3 and 27.10.4 may be strengthened: their conclusion is valid not only outside a set of zero measure in the space of coefficients but outside a proper algebraic subset of the coefficient space. To prove this strengthening, one must replace the reference to Sard's theorem by a reference to Bertini's theorem (see [Hat, p. 179, 274]).

27.11. Curves on Algebraic Surfaces. In this subsection we give an exact statement of the Hodge index theorem.

27.11.1. Definition. A compact complex manifold of complex dimension two is called a *non-singular algebraic surface* if it can be embedded into some complex projective space.

***27.11.2. Example.** In n-dimensional projective space consider a system of $n - 2$ homogeneous polynomial equations. If the differentials of these equations are independent at all the roots of the system, then the set of roots of the system constitutes a non-singular algebraic surface.

27.11.3. Definition. A *non-singular effective curve* on an algebraic surface is a one-dimensional complex submanifold on it. An *effective curve* is by definition a singular one-dimensional submanifold, i.e. a closed subset on the surface which, in the neighborhood of each point, except *a* finite number of (singular) points, is a one-dimensional submanifold.

27.11.4. To any two effective curves Γ_1 and Γ_2 on the surface we can assign the integer $\langle \Gamma_1, \Gamma_2 \rangle$ called the *intersection index* of the curves Γ_1 and Γ_2. A detailed definition of the intersection index may be found in the book [Mum]. Let us point out the following properties of the index.

27.11.5. Properties. 1) Symmetry: $\langle \Gamma_1, \Gamma_2 \rangle = \langle \Gamma_2, \Gamma_1 \rangle$.

2) Discreteness: if the curves $\Gamma_1(a)$ and $\Gamma_2(b)$ depend continuously on the parameters a, b then their intersection index is constant: $\langle \Gamma_1(a), \Gamma_2(b) \rangle = \text{const}$.

3) The intersection index of two curves which intersect transversally is equal to the number of intersection points of these curves.

In a majority of cases properties 1)–3) are sufficient for the computation of the intersection index.

27.11.6. Example. The self-intersection index of a curve of degree m on the projective plane is equal to m^2. Indeed, consider two copies of the same curve, given by a homogeneous polynomial equation of degree m. Let us change the coefficients of the first copy of the curve slightly. According to property 2), the intersection index of the first and second copy of the curve do not change. Changing the coefficients, we can assume that the two copies of the curve intersect transversally. The number of their intersection points, according to the Bézout theorem, is equal to m^2.

27.11.7. Formal finite linear combinations with rational coefficients of effective curves on the surface F constitute *linear space* $L(F)$.

The intersection index of effective curves can be extended by linearity to elements of the space $L(F)$. Then for a connected non-singular algebraic surface F we have the following theorem.

27.11.8. Theorem (the Hodge index theorem [Mum]). *The intersection index determines a hyperbolic form on the space $L(F)$.*

27.11.9. Corollary (the Hodge inequality). For any curve Γ_1 with positive self-intersection index and any other curve Γ_2 we have the inequality

$$\langle \Gamma_1, \Gamma_2 \rangle^2 \geqslant \langle \Gamma_1, \Gamma_1 \rangle \langle \Gamma_2, \Gamma_2 \rangle.$$

The Hodge inequality automatically follows from the Hodge index theorem (see proposition 27.2.3).

27.11.10. Remark. The space $L(F)$ is infinite-dimensional. The intersection form has very large kernel $J(F)$ in the space $L(F)$; according to the Severi-Neron's theorem the kernel has finite codimension in $L(F)$. The intersection form is usually considered on the finite-dimensional space $D(F) = L(F)/J(F)$. The following statement of the Hodge index theorem is more popular. The intersection index determines a non-degenerate hyperbolic form in the finite-dimensional space $D(F)$.

27.11.11. Remark. Two completely different proofs of the Hodge index theorem are known. One proof uses complex analysis and topology. It is based on the existence of the Hodge decomposition in cohomology of algebraic manifolds. This proof is generalized to non-algebraic complex analytic surfaces possessing a Kahler metric. The other proof s algebraic. It is based on a purely algebraic technique and can be generalized to algebraic surfaces defined over algebraically closed fields (and not only over the field of complex numbers). A complex-analytic proof of the Hodge theorem may be found in the book [Wel], the algebraic proof in the book [Mum].

27.12. Toric Compactification of Spaces. In this subsection we give the necessary information on toric compactification without proof. Detailed definitions and proofs may be found in the article [Kh 1].

27.12.1. Projective space $\mathbb{C}P^n$ is a compact manifold containing $\mathbb{C}^n$. It possesses the following remarkable property: the closure in $\mathbb{C}P^n$ of an $(n-k)$-dimensional manifold defined in $\mathbb{C}^n$ by the general system of polynomial equations $f_1 = \cdots = f_k = 0$ of degrees $m_1, \ldots, m_k$ is a non-singular compact $(n-k)$-dimensional submanifold in $\mathbb{C}P^n$. In this subsection we consider the toric compactification of the space $(\mathbb{C} - 0)^n$. The toric compactification is constructed on the basis of polyhedra with integer vertices $\Delta_1, \ldots, \Delta_k$. It possesses a similar remarkable property with respect to $(n-k)$-dimensional submanifolds in $(\mathbb{C} - 0)^n$ determined by Δ-non-degenerate systems of equations $f_1 = \cdots = f_k = 0$ with Newton polyhedra $\Delta_1, \ldots, \Delta_k$.

27.12.2. Definition. The finite-generated *semigroup of integer polyhedra* is the set Σ of all polyhedra of the form $\Delta = \sum_i k_i \Delta_i$ (where the k_i are non-negative integers and $\{\Delta_i\}$ is a finite family of basis polyhedra) supplied with the Minkowski addition operation.

To any finite-generated semigroup Σ we can relate the torical compactification M_Σ of the space $(\mathbb{C} - 0)^n$; its construction is described in the book [Wel].[7] It shall not need the actual construction of the compactification M_Σ but only some of the properties of M_Σ, which we give here without proof.

27.12.3. Properties. 1) M_Σ is a non-singular n-dimensional compact complex analytic manifold.

2) The manifold M_Σ contains $(\mathbb{C} - 0)^n$. The complement $M_\Sigma - (\mathbb{C} - 0)^n$ is the union of a finite number of "hypersurfaces at infinity" O_i, $M_\Sigma - (\mathbb{C} - 0)^n = \bigcup O_i$. All the hypersurfaces O_i are non-singular and intersect each other transversally.

3) The manifold M_Σ may be imbedded into some multidimensional projective space.

27.12.4. Compactification theorem (see [Kh 1]). *Suppose the* $(n-k)$-*dimensional manifold* X *is defined in* $(\mathbb{C} - 0)^n$ *by a* Δ-*non-degenerate system of equations* $f_1 = \cdots = f_k$ *whose Newton polyhedra are contained in the semigroup* Σ. *Then the closure* $\bar{X}$ *of the manifold* X *in the compactification* M_Σ *is a non-singular manifold transversally intersecting all the "surfaces at infinity"* O_i.

We shall need the following more special

27.12.5. Proposition. Suppose Γ_1 and Γ_2 are the closures in M_Σ of two curves A_1 and A_2 defined in $(\mathbb{C} - 0)^n$ by a Δ-non-degenerate system of equations whose Newton polyhedra are contained in Σ. The curves Γ_1 and Γ_ε have an intersection point at infinity (i.e. $\Gamma_1 \cap \Gamma_2 \neq A_1 \cap A_2$) if and only if the curves A_1 and A_2 have common asymptotic lines.

The statement can easily be derived from the construction of the manifold M_Σ (see [Kh 1]).

Let us state without proof one other general theorem concerning Newton polyhedra.

27.12.6. Theorem ([Kh 2]). *A manifold determined in* $(\mathbb{C} - 0)^n$ *by a* Δ-*non-degenerate system of equations* $f_1 = \cdots = f_k = 0$ *with Newton polyhedra* $\Delta_1, \ldots, \Delta_k$ *is connected under the condition that the number* k *of equations is strictly less than the number unknowns* n *and all the Newton polyhedra* Δ_i *have complete dimension, i.e.* $\dim \Delta_i = n$.

27.13. Algebraic Proof of the Alexandrov-Fenchel Inequality. Here we develop in more detail the algebraic proof of the Alexandrov-Fenchel inequality, whose outline was given in subsection 27.1.4.

Further $\Delta_1, \ldots, \Delta_n$ are fixed Newton polyhedra of complete dimension, Σ is the semigroup generated by these polyhedra and M_Σ is the toric compactification of the space $(\mathbb{C} - 0)^n$ related to the semigroup Σ.

27.13.1. Let us construct a certain algebraic surface F and a family of curves $\{\Gamma_f\}$ on it. We begin by constructing the surface F. Fix the Laurent polynomials

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[7] To be more precise, to the semigroup Σ we can relate many compactifications which are "sufficiently complete" for Σ. By M_Σ we understand any one of them.

$f_3, \ldots, f_n$ with Newton polyhedra $\Delta_3, \ldots, \Delta_n$ so that the system

$$f_3 = \cdots f_n = 0 \tag{12}$$

is strongly Δ-non-degenerate in $(\mathbb{C} - 0)^n$. (According to Theorem 27.10.3 almost any family of Laurent polynomials with fixed polyhedra possess a strongly Δ-non-degenerate system of equations). Denote by F the closure of the set of solutions of the system (12) in M_Σ. According to the compactification theorem 27.12.4, F is a non-singular algebraic surface. According to the theorem 27.12.6 the surface F is connected.

Define the curves Γ_f on the surface F. For any Laurent polynomial f denote by Γ_f the closure on the surface F of the set of solutions of the system $f = f_3 = \cdots = f_n = 0$.

27.13.2. Lemma. *If the polyhedron Δ is contained in the semigroup Σ, then for almost all Laurent polynomials f with Newton polyhedron Δ the set Γ_f is a non-singular curve.*

Proof. For almost all f with Newton polyhedron Δ the system $f = f_3 = \cdots = f_n = 0$ is strongly Δ-non-degenerate (see the theorem 27.7.5). According to the compactification theorem, if the system is Δ-non-degenerate, then the set Γ_f is a non-singular curve.

27.13.3. Proposition. *If the Newton polyhedra Δ_g and Δ_h of the Laurent polynomials g and h are contained in the semigroup Σ and the curves Γ_g and Γ_h on the surface F are non-singular, then the intersection index of the curves Γ_g and Γ_h is equal to $n!\,V(\Delta_g, \Delta_h, \Delta_3, \ldots, \Delta_n)$.*

Proof. We shall say that the Laurent polynomials g and h with Newton polyhedra Δ_g and Δ_h constitute a nice pair if the systems of equations

$$g = f_3 = \cdots = f_n = 0 \tag{13}$$

$$h = f_3 = \cdots = f_n = 0 \tag{14}$$

$$g = h = f_3 = \cdots = f_n = 0 \tag{15}$$

are all strongly Δ-non-degenerate in $(\mathbb{C} - 0)^n$. According to theorem 27.7.5, almost all pairs of Laurent polynomials with fixed Newton polyhedra are nice. For a nice pair g, h the curves defined in $(\mathbb{C} - 0)^n$ by the systems (13) and (14) have no common asymptotic lines (in the converse case the system (15) will be degenerate). Therefore for any nice pair g, h the curves Γ_g and Γ_h have no intersection points at infinity (see the proposition 27.12.5). Thus for a nice pair g and h the number of intersection points of the curves Γ_g and Γ_h is equal to the number of roots of the system (15) in $(\mathbb{C} - 0)^n$. According to the theorem on the number of roots, this number equals $n!\,V(\Delta_g, \Delta_h, \Delta_3, \ldots, \Delta_n)$. Since the system (15) is non-degenerate, all the intersection points of the curves Γ_g and Γ_h are simple (i.e. the curves Γ_g and Γ_h intersect transversally). Hence the intersection index of the curves Γ_g and Γ_h for a nice pair g, h equals $n!\,V(\Delta_g, \Delta_h, \Delta_3, \ldots, \Delta_n)$. If the curves Γ_g and Γ_h are non-singular, then the Laurent polynomials g and h do not necessarily constitute a nice pair (for example, if $g = h$), then, having slightly changed the coefficients of Laurent polynomials g and h, we can assume that the slightly changed pair has become nice. The proposition is proved entirely, since the intersection index of curves does not change in their deformation.

27.13.4. Theorem. *For polyhedra $\Delta_1, \ldots, \Delta_n$ of complete dimension whose vertices have integer coordinates the Alexandrov-Fenchel inequality holds.*

Proof. Consider the surface F defined at the beginning of this subsection. This surface is connected, hence we can apply the Hodge index theorem to it. On the surface F consider a pair of non-singular curves Γ_{f_1} and Γ_{f_2}, where f_1 and f_2 are Laurent polynomials with Newton polyhedra Δ_1 and Δ_2. The curve Γ_{f_1} has a positive self-intersection index on the surface F: according to our statement $\langle \Gamma_{f_1}, \Gamma_{f_2} \rangle = n!\, V(\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n)$, while the mixed volume of a body of complete dimension is positive. (Obviously, the curve Γ_{f_2} also has a positive self-intersection index). Hence we can apply the Hodge inequality $\langle \Gamma_{f_1}, \Gamma_{f_2} \rangle^2 \geqslant \langle \Gamma_{f_1}, \Gamma_{f_1} \rangle \langle \Gamma_{f_2}, \Gamma_{f_2} \rangle$. Substituting it into the inequality for the value of the intersection indices and self-intersection indices of the curves Γ_{f_1} and Γ_{f_2}, we obtain

$$V^2(\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n) \geqslant V(\Delta_1, \Delta_1, \Delta_3, \ldots, \Delta_n) V(\Delta_2, \Delta_2, \Delta_3, \ldots, \Delta_n).$$

The theorem is proved.

27.13.5. Corollary. *The Alexandrov-Fenchel inequality holds for arbitrary convex bodies.*

Proof. It follows from the theorem that the Alexandrov-Fenchel theorem holds for polyhedra of complete dimension with vertices at rational points (by a change of units we can always assume that the vertices are located at integer points). To complete the proof, it remains to refer to the continuity of mixed volume.