INDEX OF A POLYNOMIAL FIELD

A. G. Khovanskii

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In this article a number of estimates are given for the index of a polynomial vector field with components of fixed degrees. Examples are given showing that these estimates are best possible.

The proof of the estimates in the nondegenerate case is closely related to the proof in Petrovskii and Oleinik [1], where the Euler characteristics of some algebraic sets are estimated. In addition, the proof is closely related to the proof in the recent article by Arnol'd [2] and clarifies the connection between these two methods of argument. As in [2], the index is associated with the signature of a certain quadratic form (see also [3, 4]). As in [1], a key factor in the proof is the use of the Euler–Jacobi formula.

I am grateful to V. I. Arnol'd for informing me of his results prior to the publication of [2] and for interesting me in this subject. Arnol'd posed the problem (see [2]) of whether the Petrovskii–Oleinik estimates are best possible. This question is answered affirmatively in this article.

1. Notation and Statement of the Results

1.1. Notation. Let $V = P_1, \ldots, P_n$ be a vector field in R^n with polynomial components P_i, and let P_0 be a polynomial. We denote by ind the sum of the indices of all singular points of the field V in R^n, and denote by ind^+ and ind^- the sum of the indices of the singular points of the field V in the regions where $P_0 > 0$ and $P_0 < 0$. We will say that the pair V, P_0 has degree not exceeding (equal to) m, m_0, where $m = m_1, \ldots, m_n$, provided the degrees of all the polynomials P_i, $i = 0, \ldots, n$, do not exceed (are equal to) m_i. We will say that the pair V, P_0 is nondegenerate if, first of all, the real hypersurface $P_0 = 0$ does not pass through the singular points of V and, second, the real singular points of the field V have multiplicity one and "lie in the finite part of the space R^n." (We write the last condition out in more detail. Let the $\overline{P}_i$ be homogeneous polynomials of degree m_i in the variables $x_0, x_1, \ldots, x_n$, such that $\overline{P}_i(1, x_1, \ldots, x_n) \equiv P_i(x_1, \ldots, x_n)$. The last condition means that the system $\overline{P}_1 = \ldots = \overline{P}_n = x_0 = 0$ has only the trivial solution $x_0 = x_1 = \ldots = x_n = 0$.)

We introduce some notation.

$\Delta(m)$ is the parallelepiped in R^n defined by the inequalities $0 \leqslant y_1 \leqslant m_1 - 1, \ldots, 0 \leqslant y_n \leqslant m_n - 1$.

$\mu = m_1 \cdot \ldots \cdot m_n$ is the number of integer points in the parallelepiped $\Delta(m)$.

$\Pi(m)$ is the number of integer points in the central section $y_1 + \ldots + y_n = {}^1/_2(m_1 + \ldots + m_n - n)$ of $\Delta(m)$.

$\Pi(m, m_0)$ is the number of integer points of $\Delta(m)$ satisfying the inequalities

$$\frac{1}{2}(m_1 + \ldots + m_n - n - m_0) \leqslant y_1 + \ldots + y_n \leqslant \frac{1}{2}(m_1 + \ldots + m_n - n + m_0).$$

$O(m, m_0)$ is the number of integer points of $\Delta(m)$ satisfying the inequalities

$$\frac{1}{2}(m_1 + \ldots + m_n - n - m_0) \leqslant y_1 + \ldots + y_n \leqslant \frac{1}{2}(m_1 + \ldots + m_n - n).$$

We note that $O(m, m_0) = {}^1/_2(\Pi(m, m_0) + \Pi(m))$ and that $\Pi(m) \equiv \Pi(m, m_0) \equiv \mu \bmod 2$.

1.2. Statement of the Results. THEOREM 1. If V, P_0 is a nondegenerate pair of degree m, m_0, the numbers $a = \mathrm{ind}$, $b = \mathrm{ind}^+ - \mathrm{ind}^-$ and $c = \mathrm{ind}^+$, satisfy the inequality $|a| \leqslant \Pi(m)$, $|b| \leqslant \Pi(m, m_0)$, and $|c| \leqslant O(m, m_0)$ and the congruences $a \equiv b \equiv \mu \bmod 2$. Conversely,

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for every number a (number b, number c) satisfying these conditions, there exists a nondegenerate pair V, P_0 of degree m, m_0 for which ind $= a$ ($\text{ind}^+ - \text{ind}^- = b$, $\text{ind}^+ = c$).

COROLLARY 1. The index ind of an isolated singular point of the field $V = P_1, \ldots, P_n$ with homogeneous components of degree $m = m_1, \ldots, m_n$ satisfies the inequality $|\text{ind}| \leqslant \Pi(m)$ and the congruence ind $\equiv \mu \bmod 2$. The number ind is not subject to any other restrictions.

The inequality $|\text{ind}| \leqslant \Pi(m)$ figuring in Corollary 1 was proved by Arnol'd [2] and called by him the Petrovskii–Oleinik inequality. Corollary 1 also asserts that this inequality is best possible.

The index for vector fields V with "singular points at infinity" can also be estimated. The number ind^+ is defined if the region $P_0 > 0$ contains only isolated singular points of the field V. The number ind is defined if all the singular points of V are isolated.

THEOREM 2. Assume that the number ind^+ is defined for a pair V, P_0 of degree not exceeding m, m_0. Then if $m_0 + \ldots + m_n \equiv n \bmod 2$, the absolute value of ind^+ does not exceed $O(m, m_0)$. In this case there exist no other restrictions on the number ind^+. If $m_0 + \ldots + m_n \not\equiv n \bmod 2$ and m_0 is even, then the absolute value of ind^+ does not exceed $O(m, m_0 + 1)$. In this case there exist pairs V, P_0 for which $\text{ind}^+ = \pm O(m, m_0 + 1)$.

COROLLARY 2. Let V be a vector field of degree not exceeding $m = m_1, \ldots, m_n$, with isolated singular points. Then if $m_0 + \ldots + m_n \equiv n \bmod 2$, the estimate $|\text{ind}| \leqslant \Pi(m)$ is valid, while if $m_0 + \ldots + m_n \not\equiv n \bmod 2$ we have the estimate $|\text{ind}| \leqslant O(m, 1)$. Both of these estimates are best possible.

We give a few words concerning the organization of this material. In Sec. 3 examples are given of pairs V, P_0. These examples prove Theorems 1 and 2 in one direction, as well as their corollaries. Projective transformations, which are discussed in Sec. 2 are useful in the construction of the examples. In Sec. 4 we discuss the relation between the index and the signature of a certain quadratic form on the function algebra L_T. The algebra L_T is described in Sec. 5. In Sec. 6 the proof of Theorem 1 is completed, and the proof of Theorem 2 is finished in Sec. 7.

2. Projective Transformations

Let Γ be a hyperplane in $\mathbf{R}^n$ defined by a linear inhomogeneous equation $l(x) = l_1(x) + l_0 = 0$. We construct a projective transformation $g\colon \mathbf{R}^n \to \mathbf{R}^n$, taking the hyperplane Γ into the hyperplane at infinity; $g(x) = [1/l(x)]A(x)$, where A(x) is an affine transformation. A projective transformation of a pair V, P_0, where V is a vector field with components $P_1, \ldots, P_n$ of degrees $m_1, \ldots, m_n$ and P_0 is a polynomial of degree m_0, is a pair $\tilde{V}, \tilde{P}_0$, where $\tilde{V} = \tilde{P}_1, \ldots, \tilde{P}_n$ and $\tilde{P}_i(x) = l^{m_i}(x) P_i(g(x))$ for $i = 0, 1, \ldots, n$. If a is a singular point of the field V, then the point $\tilde{a} = g^{-1}(a)$ is singular for $\tilde{V}$. The Jacobian det $\partial g/\partial x$ is defined outside the hyperplane Γ and vanishes nowhere. We say that a transformation $x \to g(x) = [1/l(x)]A(x)$ is positive if its Jacobian is positive in the region $l(x) > 0$. For odd n, the space $\mathbf{RP}^n$ is orientable and positive transformations coincide with orientation-preserving transformations. In the general case, positive transformations correspond precisely to linear transformations of $\mathbf{R}^{n+1}$ with positive determinant.

We will be interested in how the global characteristics ind, ind^+, and $\text{ind}^+ - \text{ind}^-$ of the pair V, P_0 change under projective transformations. Each of these global characteristics is obtained by summing the corresponding local characteristics over the set X of singular points of V. This can be written symbolically as $F(V, P_0) = \sum_{a \in X} F(V, P_0)_a$, where F is one of ind, ind^+, or $\text{ind}^+ - \text{ind}^-$.

Given a field V and a projective transformation $x \to g(x)$, we write X(g) for the set of singular points a of V for which $\tilde{a} = g^{-1}(a)$ is defined. The characteristic F of the pair V, P_0 is called projectively invariant if for every positive $g = [1/l(x)]A(x)$ and $a \in X(g)$ the equality $F(V, P_0)_a = F(\tilde{V}, \tilde{P}_0)_{\tilde{a}}$ holds. The characteristic F is said to be projectively anti-invariant if $F(V, P_0)_a = \operatorname{sign} l(\tilde{a}) F(\tilde{V}, \tilde{P}_0)_{\tilde{a}}$.

A simple verification proves the following assertion.

Assertion 1. The following characteristics are projectively invariant: ind, if $m_1 + \ldots + m_n \neq n \bmod 2$; $\text{ind}^+ - \text{ind}^-$, if $m_0 + \ldots + m_n \neq n \bmod 2$; and ind^+, if $m_0 + \ldots + m_n \not\equiv n \times \bmod 2$.

2. The following characteristics are projectively antiinvariant: ind, if $m_1 + \ldots + m_n = n \bmod 2$; $\mathrm{ind}^+ - \mathrm{ind}^-$, if $m_0 + \ldots + m_n = n \bmod 2$; and ind^+, if $m_0 + \ldots + m_n = n \times \bmod 2$ and m_0 is even.

We note that the characteristic ind^+ for odd m_0 is in general neither invariant nor antiinvariant.

We write Γ_∞ for the image of the plane at infinity under a projective transformation. For $g(x) = [1/l(x)]A(x)$, where A(x) is a linear transformation and $l(x) = l_1(x) + l_0$, the equation of the hyperplane Γ_∞ has the form $l_1(A^{-1}(x)) = 1$. We associate with the hyperplane $l_1(x) = p$, where $p > 0$, the transformation $x \to (p \cdot x)/(l_1(x)+1)$, for which this plane is Γ_∞. The invariant characteristics of the singular points of the field are preserved under a projective transformation. The antiinvariant characteristics remain unchanged for the singular points lying in one of the halfspaces bounded by Γ_∞ and change sign for the singular points lying in the other halfspace.

3. Examples

3.1. In the construction of our examples, a principal role will be played by the simplest field V(m) of degree $m = m_1, \ldots, m_n$ with components $P_i = \prod_{0 \leqslant k \leqslant m_i - 1} (x_i - k), i = 1, \ldots, n$. We note that all the singular points of V(m) coincide with the integral points of the polyhedron $\Delta(m)$ defined by the inequalities $0 \leqslant x_i \leqslant m_i - 1$, $i = 1, \ldots, n$. The signs of the Jacobian at the singular points alternate in "chessboard order." Moreover, at the singular points lying on the single section $\Sigma x_i = k$, the sign of the Jacobian is constant. Upon passing to to the next section $\Sigma x_i = k + 1$, this sign is replaced by the opposite sign. In this section we often encounter the number $^1/_2\Sigma(m_i - 1)$, which we denote by ρ.

3.2. We discuss the characteristic ind in detail. Consider the case $m_0 + \ldots + m_n \not\equiv n \times \bmod 2$. In this case $\Pi(m) = 0$, and by Theorem 1 every nondegenerate field V has zero index. Thus, the field V(m) contains equal numbers of singular points on the sections $\Sigma x_i = \rho + {}^1/_2 + k$ and $\Sigma x_i = \rho - {}^1/_2 - k$, the signs of the Jacobians at the singular points in these sections being opposite. The characteristic ind in this case is invariant. We construct a projective transformation such that the plane Γ_∞ has the equation $\Sigma x_i = \rho - {}^1/_2$. The absolute value of the index is equal to $O(m, 1)$ for the degenerate field V so obtained: Indeed, the inverse images of the singular points on the section $\Sigma x_i = \rho - {}^1/_2$ "lie at infinity" and the inverse images of the singular points on the section $\Sigma x_i = \rho + {}^1/_2$ are not cancelled, and there are $O(m, 1)$ of them. In order to construct an example of a field V with index of opposite sign, it suffices to change the sign of one of the components of the field V.

We consider the case $m_1 + \ldots + m_n \equiv n \bmod 2$. In this case, the characteristic ind is antiinvariant. The central section $\Sigma x_i = \rho$ of the rectangular box $\Delta(m)$ contains exactly $\Pi(m)$ singular points of the field V(m). The sections $\Sigma x_i = \rho - k$ and $\Sigma x_i = \rho + k$ contain the same number of singular points with Jacobians of the same sign. We carry out a projective transformation $x \to g(x)$ for which the plane Γ_∞ has the equation $\Sigma x_i = \rho - {}^1/_2$, e.g., the transformation $x \to (\rho - {}^1/_2)\, x/(\Sigma x_i + 1)$. The sections $\Sigma x_i = \rho + k$ and $\Sigma x_i = \rho - k$ for $k > 0$ lie in different half spaces bounded by Γ_∞. The indices of the inverse images of the singular points of the field V(m) lying in these sections cancel one another. Therefore, the absolute value of the index of the field $\tilde{V}$ is equal to the number of singular points of V(m) lying on the section $\Sigma x_i = \rho$, i.e., it is equal to $\Pi(m)$. Here is an explicit formula for the component $\tilde{P}_i$ of the field $\tilde{V}$: $\tilde{P}_i(x) = \prod_{0 \leqslant k < m_i} \left[\left(\rho - \frac{1}{2}\right) x_i - k\left(\sum x_i + 1\right)\right]$.

The index of the field $\tilde{V}$ does not change if the position of the plane Γ_∞ is perturbed slightly. We define Γ_∞ by the equation $\Sigma a_i x_i = t$, where the a_i are scalars close to unity which are independent over the rationals, and t is close to $\rho - {}^1/_2$. We now begin to let t get bigger. This will not change the index of the field $\tilde{V}$ until Γ_∞ passes through a singular point of V(m) from the direction of the central section $\Sigma x_i = \rho$. When this occurs, $\tilde{V}$ becomes degenerate and its index changes by one. If the number t is increased a bit more, the field $\tilde{V}$ again becomes nondegenerate and its index again changes by one. Continuing the motion of the plane Γ_∞, we obtain examples of nondegenerate fields $\tilde{V}$ of degree m with any index satisfying the conditions $|\mathrm{ind}| \leqslant \Pi(m)$, $\mathrm{ind} \equiv \mu \bmod 2$. The leading homogeneous components of the field $\tilde{V}$ form a field with an isolated singular point at zero with the same index. Moving the plane as above, we again obtain examples of degenerate vector fields with arbitrary index of absolute value not exceeding $\Pi(m)$.

3.3. We turn now to the characteristics $\mathrm{ind}^+ - \mathrm{ind}^-$ and ind^+. We first discuss the case when $m_0 + \dots + m_n \equiv n \bmod 2$. The characteristic $\mathrm{ind}^+ - \mathrm{ind}^-$ in this case is anti-invariant. Assume in addition that m_0 is odd, $m_0 = 2q + 1$. We consider the pair $V(m)$, P_0, where $P_0 = \prod_{k \in I} \left(\sum x_i - \rho + k\right)$ and $I = \{\pm 1, \dots, \pm q, q + 1\}$. We carry out a projective transformation such that Γ_∞ has the equation $\Sigma x_i = \rho$. The pair $\tilde{V}$, $\tilde{P}_0$ obtained by means of this transformation will have the extremal characteristics $|\mathrm{ind}^+ - \mathrm{ind}^-| = \Pi(m, m_0)$ and $|\mathrm{ind}^+| = O(m, m_0)$. By moving the plane Γ_∞ as above, we get examples of nondegenerate and degenerate pairs $\tilde{V}$, $\tilde{P}_0$, for which the characteristics $\mathrm{ind}^+ - \mathrm{ind}^-$ and ind^+ take all values compatible with the assertions of Theorems 1 and 2.

The situation is analogous in the case of even m_0, $m_0 = 2q$, and $m_0 + \dots + m_n \equiv n \bmod 2$. Here it is necessary to begin with a pair $V(m)$, P_0, where $P_0 = (-1)^q \prod_{k \in I} \left(\sum x_i - \rho + \frac{1}{2} - k\right)$ and $I = \{\pm 1, \dots, \pm q\}$, and the planes Γ_∞ have the equation $\Sigma x_i = \rho - 1/2$.

We now turn to the case $m_0 + \dots + m_n \not\equiv n \bmod 2$. The case $m_0 = 0$ has already been analyzed, and we may assume that $m_0 > 0$. In this case the equalities $\Pi(m, m_0) = \Pi(m, m_0 - 1)$ and $O(m, m_0) = O(m, m_0 - 1)$ are valid. Therefore, in order to construct nondegenerate pairs V, P_0 with arbitrary characteristics $\mathrm{ind}^+ - \mathrm{ind}^-$ and ind^+ compatible with Theorem 1, it is enough to make use of the examples already constructed of nondegenerate pairs V, Q, where the degree of Q is equal to $m_0 - 1$. It suffices to consider pairs V, P_0 where $P_0 = Q(\Sigma x_i + a)$. For sufficiently large a, the pairs V, P_0 and V, Q have the same characteristics ind^+ and ind^-.

We give an example of a degenerate pair V, P_0 with extremal characteristic ind^+ for even $m_0 = 2q$ and $m_0 + \dots + m_n \not\equiv n \bmod 2$. We take the field $V(m)$ and the polynomial $P_0 = (-1)^q \prod_{k \in I} \left(\sum x_i - \rho - k\right)$, where $I = \{\pm 1, \dots, \pm q\}$. We carry out a projective transformation for which the plane Γ_∞ has the equation $\Sigma x_i = \rho - 1/2$. For the resulting pair $\tilde{V}$, $\tilde{P}_0$ the characteristic ind^+ has absolute value equal to $O(m, m_0 + 1)$.

4. Signature and Index

4.1. A Finite Set with Involution. Let A be a finite set containing μ elements, $\tau: A \to A$ an involution of A, and let X be the set of fixed points of τ. We consider the algebra L_τ over the field $\mathbb{R}$ consisting of all complex value functions on A for which $f \cdot \tau = \bar{f}$. Let φ be a fixed function in L_τ which is nowhere zero. The number of points of the set X at which φ is positive is denoted by φ^+, the number of points at which it is negative, by φ^-. We consider the bilinear form ω_φ on L_τ defined by $\omega_\varphi(f, g) = \sum_{a \in A} \varphi(a) f(a) g(a)$. The **signature** of a quadratic form K is denoted by σK.

LEMMA. The dimension of the algebra L_τ is equal to μ. The quadratic form $K_\varphi(f) = \omega_\varphi(f, f)$ takes real values and is nondegenerate. The signature σK_φ is equal to $\varphi^+ - \varphi^-$. In particular, σK_φ for $\varphi = 1$ is equal to the number of fixed points of τ.

Proof. Under the action of τ, the set A decomposes into invariant sets A^k consisting of one or two points. Let L_τ^k denote the subalgebra of L_τ consisting of functions with support A^k, $L_\tau = \Sigma L_\tau^k$. The subspaces L_τ^k are orthogonal with respect to the form ω_φ. If A^k consists of a single point a, $a \in X$, then $\dim L_\tau^k = 1$ and the signature of the restriction of the form K_φ to L_τ^k is equal to $\operatorname{sign} \varphi(a)$. For two-point sets $A^k = \{a, \tau a\}$ $\dim L_\tau^k = 2$. In this case, the restriction of the form K_φ to L_τ^k is equal to $\varphi(a) f^2(a) + \varphi(\tau a) f^2(\tau a) = 2\mathrm{Re}\, \varphi(a) f^2(a)$. As is easily seen, the signature of such a form is zero. The lemma is proved.

COROLLARY. 1. The congruence $\varphi^+ - \varphi^- \equiv \mu \bmod 2$ holds. 2. Let L_0 be any linear subspace of the algebra L_τ on which the form K_φ is identically equal to zero. Then the estimate $|\varphi^+ - \varphi^-| \leqslant \mu - 2 \dim L_0$ is valid. If the null subspace is maximal, this estimate is best possible.

Indeed, the signature of a nondegenerate form always has the same parity as the dimension of the space. In addition, the estimate $|\sigma K| \leqslant \mu - 2 \dim L_0$ holds for every nondegenerate form K on $\mathbb{R}^\mu$ with nullspace L_0. For a maximal subspace L_0, this estimate is an equality.

We will apply the lemma and corollary to the case when A is the set of complex singular points of a real vector field and $\tau: A \to A$ is the involution given by complex conjugation.

4.2. We consider a real vector field V in R^n with polynomial components $V = P_1, \ldots, P_n$. Let $A \subset C^n$ be the set of complex solutions of the system

$$P_1 = \ldots = P_n = 0 \tag{1}$$

and let the involution $\tau: A \to A$ be complex conjugation. The set X of fixed points under τ coincides with the set of real solutions of system (1). We assume that all the complex solutions $a \in A$ of system (1) have multiplicity one. This means that the Jacobian $j(x) = \det \partial P/\partial x$ of (1) does not vanish at the points of A. Let P_0 be any polynomial with real coefficients which does not vanish at the points of A.

We obtain the following assertion by applying the preceding lemma.

Assertion. The signature σK_φ of the quadratic form $K_\varphi(f) = \sum_{a \in A} \varphi(a) f^2(a)$ is equal to the number of real singular points of the field V if $\varphi = 1$. If $\varphi = 1/j$, where j is the Jacobian of system (1), then $\sigma K_\varphi = \text{ind}$, and if $\varphi = P_0/j$, then $\sigma K_p = \text{ind}^+ - \text{ind}^-$.

In order to estimate the numbers ind and $\text{ind}^+ - \text{ind}^-$, it is now necessary to describe the algebra L_τ and exhibit a nullspace for the form K_φ which is as large as possible for $\varphi = 1/j$ and $\varphi = P_0/j$. The null subspace is obtained with the aid of the Euler—Jacobi formula, which we recall.

Consider a system of n polynomial equations of degrees $m_1, \ldots, m_n$ in n complex unknowns,

$$P_1 = \ldots = P_n = 0.$$

We assume that the set of roots of the system contains exactly $\mu = m_1 \cdot \ldots \cdot m_n$ elements. In this case, the Jacobian of the system $j = \det \partial P/\partial x$ does not vanish on the set A. Then for every polynomial Q of degree less than $\Sigma m_i - n$, we have the following Euler—Jacobi formula: $\sum_{a \in A} \frac{Q(a)}{j(a)} = 0.$

A purely algebraic proof of this formula can be found in [5]. An analytic proof and a generalization to nondegenerate systems of equations with fixed Newton polyhedra is given in [6].

5. Convenient Systems of Equations

A system of equations $P_1 = \ldots = P_n = 0$ of degrees $m_1, \ldots, m_n$ will be called nondegenerate if it has exactly $\mu = m_1 \cdot \ldots \cdot m_n$ distinct roots.

Consider the parallelepiped $\Delta(m)$ in R^n defined by the inequalities $0 \leqslant y_1 \leqslant m_1 - 1, \ldots, 0 \leqslant y_n \leqslant m_n - 1$. Let M(m) be the space of polynomials with Newton polyhedron $\Delta(m)$. A polynomial $Q \in M(m)$, if and only if the degree of Q with respect to the variables x_i is less than m_i. The dimension of the space M(m) over the field C is equal to $m_1 \cdot \ldots \cdot m_n = \mu$.

A nondegenerate system is said to be convenient if every complex valued function f on the set A of roots of the system is the restriction of some polynomial in M(m).

LEMMA 1. The system of equations $\prod_{0 \leqslant k < m_1} (x_1 - k) = \ldots = \prod_{0 \leqslant k < m_n} (x_n - k) = 0$ is convenient.

Indeed, the set A of roots of this system contains precisely $\mu = m_1 \cdot \ldots \cdot m_n$ elements. In addition, the equations of the system can be rewritten in the form of equalities $x_1^{m_1} = Q_1(x_1), \ldots, x_n^{m_n} = Q_n(x_n)$, in which $Q_1, \ldots, Q_n$ are polynomials of degrees $m_1 - 1, \ldots, m_n - 1$. Using these equations, it is not hard to show that every polynomial Q(x) coincides on the set A with some polynomial in the space M(m). This implies Lemma 1, since every function f on the finite set A is the restriction of some polynomial.

LEMMA 2. The inconvenient systems form a hypersurface in the space of all systems of degree m.

Indeed, as is well known, the degenerate systems form a hypersurface in the space of all systems of degree m. Take any nondegenerate system and enumerate its roots arbitrarily,

$a_1, \ldots, a_\mu$. Then enumerate in some way the integer points in the Newton polyhedron $\Delta(m)$. These numerations define bases in the μ-dimensional space $C(A)$ of all complex-valued functions on A and in the μ-dimensional space $M(m)$. Let det denote the determinant of the matrix of the restriction mapping $i\colon M(m) \to \mathbf{C}(A)$ with respect to these bases. The number $\det^2$ does not depend on the choice of enumeration; it depends only on the coefficients of the system, this dependence being analytic. A nondegenerate system is convenient if and only if the number $\det^2$ for it is distinct from zero. The function $\det^2$ is not identically equal to zero. Indeed, by Lemma 1, there exist convenient systems of degree m. Lemma 2 is proved.

Let $M(m, R)$ denote the space of polynomials with real coefficients with Newton polyhedron $\Delta(m)$, $M(m, \mathbf{R}) = M(m) \cap \mathbf{R}[x]$.

LEMMA 3. For a convenient system of equations of degree m with real coefficients, the restriction of polynomials in the space $M(m, R)$ to the set A defines an isomorphism of $M(m, R)$ with the algebra L_τ onto A, where $\tau\colon A \to A$ is the involution of complex conjugation.

The restrictions of polynomials in $M(m, R)$ clearly lie in the algebra L_τ. In addition, for convenient systems a nonzero polynomial in $M(m)$ corresponds to a nonzero function on A. Lemma 3 follows from the inclusion $M(m, \mathbf{R}) \subset M(m)$ and the fact that $\dim M(m, \mathbf{R}) = \mu$ and $\dim L_\tau = \mu$.

6. Inequalities for Nondegenerate Fields

We conclude the proof of Theorem 1 (see Sec. 1). Let V, P_0 be a nondegenerate pair of degree m, m_0. Under a small change in the coefficients of the components $P_1, \ldots, P_n$ of V and polynomial P_0, the numbers ind, ind^+, and ind^- will not change. It can therefore be assumed, without loss of generality, that $P_1 = \ldots P_n = 0$ is a convenient system of degree m (see Lemma 2, Sec. 5) and that the surface $P_0 = 0$ in C^n does not intersect the set of roots A of the system. By Lemma 3 in Sec. 5, every function on A in L_τ, where the involution τ: $A \to A$ is complex conjugation, is the restriction of a unique polynomial in $M(m, R)$. We consider the quadratic form $K_\varphi(f)$ on L_τ with $\varphi = P_0/j$ (here $j = \det \partial P/\partial x$). According to the assertion in Sec. 4, $\sigma K_\varphi = \text{ind}^+ - \text{ind}^-$. It follows from the Euler–Jacobi formula that for all polynomials f of degree less than $\frac{1}{2}\left[\sum_{i>0}(m_i - 1) - m_0\right]$, the identity $K_\varphi(f) = 0$ is valid. In our case, the inequality $|\sigma K_\varphi| \leqslant \mu - 2 \dim L_0$ takes the form $|\text{ind}^+ - \text{ind}^-| \leqslant \Pi(m, m_0)$. Indeed, μ is equal to the number of integer points in the polyhedron $\Delta(m)$, and $\dim L_0$ is equal to the number of integer points in $\Delta(m)$ satisfying the inequality $\sum y_i < {}^1/_2\left[\sum_{i>0}(m_i - 1) - m_0\right]$. The inequality $|\text{ind}^+ - \text{ind}^-| \leqslant \Pi(m, m_0)$ has been proved. The inequality $|\text{ind}| \leqslant \Pi(m)$ is a particular case of this inequality for $m_0 = 0$. Combining the inequalities $|\text{ind}| = |\text{ind}^+ + \text{ind}^-| \leqslant \Pi(m)$ and $|\text{ind}^+ - \text{ind}^-| \leqslant \Pi(m, m_0)$, we get $|\text{ind}^+| \leqslant {}^1/_2 [\Pi(m, m_0) + \Pi(m)] = O(m, m_0)$. The congruences $\text{ind} \equiv \text{ind}^+ - \text{ind}^- \equiv \mu \bmod 2$ are almost obvious (see Corollary in Sec. 4). Theorem 1 is proved (the necessary examples of pairs V, P_0 are given in Sec. 3).

7. Inequalities for Degenerate Vector Fields

We conclude the proof of Theorem 2 (see Sec. 1).

LEMMA 1. Assume that the characteristic ind^+ is defined for a pair V, P_0 of degree m, m_0. Then: 1) if the region $P_0 > 0$ is compact, $|\text{ind}^+| \leqslant O(m, m_0)$; 2) if $2k > m_0$, $|\text{ind}^+| \leqslant O(m, 2k)$.

Indeed, the first assertion of the lemma is easily reduced to Theorem 1. We prove the second assertion. The domain U_ε defined by the inequality $P_0 - \varepsilon r^{2k} > 0$, where $r^2 = \Sigma x_i^2$ and ε is small, contains the same singular points of the field V as the region $P_0 > 0$. The region U_ε is already compact, and therefore the second assertion also reduces to Theorem 1.

COROLLARY 1. 1. Assume in hypothesis 2) of Lemma 1 that m_0 is odd and $\sum_{i>0} m_i \not\equiv n \bmod 2$. Then $|\text{ind}^+| \leqslant O(m, m_0)$.

2. Under assumption 2) of Lemma 1, let m_0 be even and $\sum_{i\geqslant 0} m_i \not\equiv n \bmod 2$. Then $|\text{ind}^+| \leqslant O(m, m_0 + 1)$.

Indeed, in case 1, $O(m, m_0+1) = O(m, m_0)$, and in case 2, $O(m, m_0+2) = O(m, m_0+1)$. Corollary 1 therefore follows from Lemma 1. The examples in Sec. 2 show that the inequalities in Corollary 1 are best possible. It remains for us to consider the case of even m_0 with $\Sigma m_i \equiv n \bmod 2$.

Consider the strip U_0 in R^n contained between two parallel hyperplanes Γ_1 and Γ_2.

LEMMA 2. Assume that for every nondegenerate pair of degree m, m_0 the inequality $|F(V, P_0)| \leqslant Q(m, m_0)$ is valid and the characteristic F is antiinvariant. Then for every strip U_0 the inequality $|\sum_{a\in U_0} F(V, P_0)_a| \leqslant Q(m, m_0)$ is valid.

Proof. The planes Γ_1 and Γ_2 divide R^n into three regions U_0, U_1 and U_2. We put $c_i = \sum_{a\in U_i} F(V, P_0)_a$ for i = 0, 1, 2. It can be assumed that the planes Γ_1 and Γ_2 do not pass through the singular points of the field V (otherwise the strip U_0 can be taken to be slightly smaller). We consider two projective transformations g^1 and g^2 for which the planes Γ_∞ are Γ_1 and Γ_2. We obtain two nondegenerate pairs V^1, P_0^1 and V^2, P_0^2 of degree m, m_0, for which the absolute values of the characteristic F are equal to $|c_0 + c_1 - c_2|$ and $|c_0 - c_1 + c_2|$. By hypothesis, $|c_0 + c_1 - c_2| \leqslant Q(m, m_0)$, $|c_0 - c_1 + c_2| \leqslant Q(m, m_0)$, whence $|c_0| \leqslant Q(m, m_0)$. Lemma 2 is proved.

LEMMA 3. Under the hypotheses of Lemma 2, the inequality $|F(V, P_0)| \leqslant Q(m, m_0)$ is valid for any pair V, P_0 of degree m, m_0 for which the characteristic F is defined and such that the surface $P_0 = 0$ does not pass through singular points of the field V.

Proof. We consider a family of pairs V^t, P_0^t depending algebraically on the parameter t, $0 \leqslant t \leqslant 1$, such that the pairs V^t, P_0^t are nondegenerate for $t < 1$ and such that when $t = 1$ the pair V^1, P_0^1 coincides with the pair V, P_0. The singular points a^t of the field V^t move along algebraic curves as t varies, and at t = 1 certain of the curves a^t will tend to the singular points of the field V. The remaining curves will go off to infinity. For almost every linear function l, the number $l(a^t)$ for such curves will also tend to infinity. Let l be such a linear function and let the number p be so large that the strip U_0 defined by the inequality $|l(x)| < p$ contains all the singular points of the field V. Then for values of t close to unity, $\sum_{a^t\in U_0} F(V^t, P_0^t) = F(V, P_0)$. Indeed, the singular points which go off to infinity give no contribution to either the left or the right hand side of this equality. In addition, by virtue of the additivity of the index of a vector field $F(V, P_0)_a = \Sigma F(V^t, P_0)_{a^t}$, where the summation is over the points a^t tending to a. In order to complete the proof, it remains to apply Lemma 2.

COROLLARY 2. Under the hypothesis of Lemma 1, assume that m_0 is even and $\Sigma m_i \equiv n \bmod 2$. Then $|\mathrm{ind}^+| \leqslant O(m, m_0)$. In particular, for $m_0 = 0$ we have the inequality $|\mathrm{ind}| \leqslant \Pi(m)$.

Indeed, in the case of even m_0 and $\Sigma m_i \equiv n \bmod 2$, the characteristic ind^+ is antiinvariant. It is easy to get rid of the extra assumption that the surface $P_0 = 0$ contains no singular points of the field V. It suffices to consider the smaller surface $P_0 - \varepsilon > 0$ for small ε.

Corollaries 1 and 2 in conjunction with the examples of Sec. 3 completely prove Theorem 2.

8. Remarks

8.1. What values can the index take for a polynomial vector field in the ball $R^2 - \Sigma x_i^2 \geqslant 0$? The following assertion answers this question for half of the degrees.

Assertion. Let V = $P_1, \ldots, P_n$ be a polynomial field of degree m = $m_1, \ldots, m_n$ which has only isolated singular points in the ball $R^2 - \Sigma x_i^2 \geqslant 0$. If $m_1 + \ldots + m_n \not\equiv n \bmod 2$ then the total index summed over these singular points can be equal to any number from $-O(m, 1)$ to $+O(m, 1)$.

Indeed, the ball $R^2 - \Sigma x_i^2 \geqslant 0$ is defined by an equation of second degree, and therefore, by Lemma 1 in Sec. 7, $|\mathrm{ind}^+| \leqslant O(m, 2)$. For $\sum_{i>0} m_i \not\equiv n \bmod 2$, $O(m, 1) = O(m, 2)$. We turn to some examples. The field V(m) (see Sec. 3) for $\sum_{i>0} m_i \not\equiv n \bmod 2$ contains exactly $O(m, 1)$ singular points on the plane $\Sigma x_i = \rho - 1/2$, and the indices of all these points have the same sign. These points can be contained in an ellipsoid which contains no other singular points. By making the ellipsoid smaller, it is possible to have any number of singular points from this plane inside the ellipsoid. Since the ellipsoid can be taken into a sphere by means of an affine transformation, the assertion is completely proved.

I do not know a precise estimate for the index of a polynomial field in a ball for $m_1 + \ldots + m_n \equiv n \bmod 2$.

8.2. Let V, P_0 be a pair of degree m, m_0 and $V = P_1, \ldots, P_n$. We consider a field $\overline{V} = \overline{P}_0, \ldots, \overline{P}_n$ in n + 1 variables $x_0, \ldots, x_n$ in which the $\overline{P}_i$ are homogeneous polynomials of degree m_i such that $\overline{P}_i(1, x_1, \ldots, x_n) = P_i(x_1, \ldots, x_n)$.

Assertion. The field $\overline{V}$ has an isolated singular point at zero in the space $\mathbf{R}^{n+1}$, if the pair V, P_0 is nondegenerate. If in addition $m_0 + \ldots + m_n \not\equiv n \bmod 2$, then the index of this point is equal to the characteristic $\mathrm{ind}^+ - \mathrm{ind}^-$ of the pair V, P_0.

We will not dwell on the proof, which is not complicated. We note that the above assertion explains the projective invariance of the characteristic $\mathrm{ind}^+ - \mathrm{ind}^-$ in the case $m_0 + \ldots + m_n \equiv n \bmod 2$. In this case, it also reduces Theorem 1 to the Petrovskii–Oleinik inequality for the index of a singular point of a homogeneous field. I do not know if there exists a similar reduction when $m_0 + \ldots + m_n \not\equiv n \bmod 2$ and $m_0 > 0$.

8.3. In their fundamental paper [1], Petrovskii and Oleinik estimate the Euler characteristics of algebraic sets. Here is an example of such an estimate.

Assertion. Let $P: \mathbf{R}^n \to \mathbf{R}$ be a polynomial of degree k such that the surface P = 0 is nonsingular and the regions P < c are compact for every $c, c \in \mathbf{R}$. Then the Euler characteristic χ of the region P < 0 satisfies the inequality $|1 - 2\chi| \leqslant \Pi(m, k-1)$, where $m = \underbrace{k-1, \ldots, k-1}_{n \text{ times}}$.

Proof. We consider the pair consisting of the field V = grad P of degree m and the polynomial P of degree k. By Morse theory, $\chi = \mathrm{ind}^+$ and $1 = \chi(\mathbf{R}^n) = \mathrm{ind}^+ + \mathrm{ind}^-$. In addition, the polynomials $-P$ and $Q = -P + (1/k)\sum x_i P'_{x_i}$ coincide at the singular points of V, and therefore the pairs V, $-P$ and V, Q have the same characteristics $\mathrm{ind}^\pm$. By Euler's formula for homogeneous functions, the polynomial Q has degree k − 1. Making use now of Theorem 1, we get $|\mathrm{ind}^+ - \mathrm{ind}^-| \leqslant \Pi(m, k-1)$, and taking into account the equality $\mathrm{ind}^+ + \mathrm{ind}^- = 1$ and $\chi = \mathrm{ind}^+$, we obtain the required inequality $|1 - 2\chi| \leqslant \Pi(m, k-1)$.

We note that this inequality for the Euler characteristic is only known to be best possible for n = 2 (see [7]).

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