THE REPRESENTABILITY OF ALGEBROIDAL FUNCTIONS BY SUPERPOSITIONS OF ANALYTIC FUNCTIONS AND ALGEBROIDAL FUNCTIONS OF ONE VARIABLE

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In this note we give a necessary condition for the representability of an algebroidal function in the form of superpositions of analytic functions and algebroidal functions of one variable. In particular, we show that an algebroidal function of two variables, $z(a, b)$, defined by the equation

$$z^5 + az + b = 0,$$

cannot be represented in the form of such superpositions in any neighborhood of the origin.

Definition 1. An analytic germ f_a, defined at a point a of a neighborhood $U \subseteq C^n$, is called algebroidal in this neighborhood if it satisfies some irreducible equation

$$f^n + p_1 f^{n-1} + \ldots + p_n = 0, \tag{1}$$

where p_i, $i = 1, \ldots, n$, are functions which are analytic in the neighborhood U.

Denote the set of all solutions of Eq. (1) in a neighborhood $V \subseteq U$ by $f(V)$.

Definition 2. Two analytic germs f_a and g_b, defined at the points a and b, respectively, of a neighborhood $U \subseteq C^n$, are called equivalent in this neighborhood, $f_a \underset{U}{\sim} g_b$, if the germ g_b can be obtained from the germ f_a by continuation along some curve lying in U.

Denote the value of the germ f_a at the point a by $f(a)$.

Denote the closure of the set of values assumed at points of a region $V \subseteq U$ by germs g_b which are equivalent to f_a in the neighborhood U by $f(V)$.

We give another definition of an algebroidal germ.

Definition 3. An analytic germ f_a defined at a point a of a neighborhood $U \subseteq C^n$, is called algebroidal in the neighborhood U if the following conditions are satisfied: 1) there exists an analytic set $M \subsetneq U$ such that the germ f_a can be analytically continued along any curve lying in U, beginning at the point a, which intersects the set M, if at all, only at the initial point; 2) at every point $b \in U$ there exist only a finite number of germs which are equivalent to the germ f_a; 3) for every compact $W \subset U$, the set $f(W)$ is compact.

Any set M which satisfies the first condition is called a prohibited set for the germ f_a.

We will show that Definitions 3 and 1 are equivalent. Suppose that a germ f_a satisfies Definition 1, and let D be the discriminant set of Eq. (1). It is easy to see that the germ f_a satisfies Definition 3 with prohibited set M.

First consider the case $a \notin M$. Let f_{ia}, $i = 1, \ldots, n$, $f_{1a} = f_a$, be the set of all germs which are equivalent on U to the germ f_a at the point a. Using Riemann's theorem on removable singularities, it is easy to see that the symmetric functions of the germs f_{ia},

$$p_1 = -\sum_{i=1}^{n} f_{ia}, \ldots, \quad p_n = (-1)^n f_{1a} \ldots f_{na}$$

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are germs of functions which are analytic in the neighborhood U. The germ f_a satisfies the equation

$$f^n + p_1 f^{n-1} + \ldots + p_n = 0.$$

Now consider the case $a \in M$. We take, together with the germ f_a a germ g_b which is equivalent to the germ f_a on the neighborhood U at a point b $\notin$ M. It is clear that the germ f_a satisfies the same equation as the germ g_b.

Definition 4. Let f_a be a non-constant algebroidal germ satisfying Eq. (1). Denote the set of roots of the equation

$$b^n + p_1(x) b^{n-1} + \ldots + p_n(x) = 0$$

by $f^{-1}(b)$.

Definition 5. A point a is called regular for the irreducible equation

$$y^n + p_1 y^{n-1} + \ldots + p_n = 0,$$

where the p_i are functions which are analytic in the neighborhood U, if at this point there exist exactly n distinct germs which satisfy this equation. A point a is called regular for the algebroidal germ f_a in a neighborhood U if there exists a prohibited set $M \subset U$ which does not contain the point a.

Definition 6. A set of k analytic germs at the point a, given in a definite order $\mathfrak{f}_a = (f_{1a}, \ldots, f_{ka})$, is called a vector-germ of dimension k at the point a. All of the preceding definitions can be generalized to vector-germs in a unique way.

LEMMA 1. Let $\mathfrak{f}_a$ be an algebroidal k-dimensional vector-germ in a neighborhood $U \subseteq C^n$, and let g be an analytic function in a region $V \subseteq C^k$ such that $V \supseteq \mathfrak{f}(U)$. Then 1) the germ $g \circ f_a$ is algebroidal in the neighborhood U; 2) there exists an analytic set $M \subset U$ such that if the germ $p_b \underset{U}{\sim} g \circ \mathfrak{f}_a$ and the point b lies in $U \setminus M$, then $p_b = g \circ \varphi_b$, where φ_b is a germ which is equivalent to $\mathfrak{f}_a$ on the neighborhood U.

Proof. As the prohibited set M for the germ $g \circ \mathfrak{f}_a$ it suffices to take any prohibited set of $\mathfrak{f}_a$. It will satisfy the condition of the second assertion in the lemma as well.

LEMMA 2. Let f_a be an algebroidal germ in a neighborhood $U \subseteq C^n$ which is not equal to a constant, and let g_c be an algebroidal germ in a neighborhood $V \subseteq C^1$ such that $V \supseteq f(U)$ and $c = f(a)$. Then 1) the germ $g_c \circ f_a$ is algebroidal in the neighborhood U; 2) there exists an analytic set $M \subset U$ such that if the germ $p_b \underset{U}{\sim} g_c \circ f_a$ and the point b lies in $U \setminus M$, then $p_b = q_d \circ \varphi_b$, where $d = \varphi(b)$, φ_b is a germ which is equivalent to f_a on the neighborhood U, and q_d is a germ which is equivalent to g_c on the neighborhood V.

Proof. Let D(x) be the discriminant of the irreducible equation satisfied by g_c. By Lemma 1, the germ $D \circ f_a$ is algebroidal in the neighborhood U. It is easy to verify that $D \circ f_a$ is not identically equal to a constant. Let M_1 denote the set $(D \circ f)^{-1}(0)$ (see Definition 4) and M_2 some prohibited set of the germ f_a. As a prohibited set M for the germ $g_c \circ f_a$ we can take the set $M_1 \cup M_2$. It will also satisfy the condition contained in the second assertion of the lemma.

We introduce some more notation.

Denote the ring of analytic germs on a linearly connected set $X \subseteq C^n$ by H^n_X. This ring is an integral domain. Denote the corresponding field of quotients by L^n_X.

Let $\mathfrak{f}_a = (f_{1a}, \ldots, f_{ka})$ be a vector-germ which is algebroidal in a neighborhood $U \subseteq C^n$, and let $P_i(f_i, H^n_U) = 0$, $i = 1, \ldots, k$, be the irreducible equations of its components. The expanded field of the polynomials P_i over L^n_X will be denoted by $L^n_X(\mathfrak{f})$, and the Galois group of $L^n_X(\mathfrak{f})$ over L^n_X by $\Gamma^n_X(\mathfrak{f})$. Let $H^n_X(\mathfrak{f})$ denote the subring of the field $L^n_X(\mathfrak{f})$ spanned by H^n_X and the roots of all of the polynomials $P_i(f_i, H^n_U) = 0$. If the point a lies in X, then we can define a ring $H^n_X\langle \mathfrak{f}_a \rangle$ which is the subring of H^n_a spanned by the ring H^n_X and the germs $f_{1a}, \ldots, f_{ka}$.

It is easy to see that if a is a regular point of the germ f_b, $f_{1a}, \ldots, f_{ka}$ is the set of all germs at the point a which are equivalent to f_b on the neighborhood U, and X is a linearly connected set containing the point a, then

$$H^n_X \langle f_{1a}, \ldots, f_{ka} \rangle = H^n_X(f).$$

Before formulating Lemma 3, we note that if the equation $y^k + p_1 y^{k-1} + \ldots + p_k = 0$, where the p_i are analytic functions in a neighborhood V containing the point a is irreducible over the ring H_a^n, then all k roots of this equation coincide at the point a.

LEMMA 3. Let $P_i(y_i, H_a^n) = 0$, $i = 1, \ldots, k$, be integral irreducible equations over the ring H_a^n; let $b_i = y_i(a)$ be the values of the functions y_i at a; let $b = (b_1, \ldots, b_k)$ be a point in the space C^k; and let $g \in H_b^k$ be an analytic germ at $b \in C^k$. Then the point a has a neighborhood V such that

1) the equations $P_i(y_i, H_a^n) = 0$ can be extended to equations $P_i(y_i, H_V^n) = 0$ in this neighborhood, and the function g can be defined in a neighborhood U containing y(V);

2) for every point d in V and every vector-germ $\mathbf{y}_d$ satisfying the equations $P_i(y_{id}, H_V^n) = 0$, the ring $H_V^n\langle g \circ \mathbf{y}_d\rangle$ is contained in $H_V^n\langle \mathbf{y}_d\rangle$.

Proof. Take the point (a, b) in the space $C^n \times C^k$. In the ring $H_{(a,b)}^{n+k}$ consider the germs $P_i(y_i, H_a^n)$ and the germ g^* obtained from g under the projection $\pi: C^n \times C^k \to C^k$. Choose a neighborhood V of a in in the space C^n which is small enough for successive application of Weierstrass' theorem on the partition of germs. Applying this theorem, we get

$$g \circ \mathbf{y}_d = \sum_{|i|<m} p_i \mathbf{y}_d^i,$$

where $i = (i_1, \ldots, i_k)$, $|i| = i_1 + \ldots + i_k$; $\mathbf{y}_d^i = y_{1d}^{i_1} \ldots y_{kd}^{i_k}$; $p_i \in H_V^n$; and m is some integer. Therefore $H_V^n\langle g \circ \mathbf{y}_d\rangle \subseteq H_V^n\langle \mathbf{y}_d\rangle$.

LEMMA 4. Let $P_1(y, H_a^n) = 0$ and $P_2(z, H_b^1) = 0$ be integral irreducible equations over the corresponding rings, where $y(a) = b$ and y is not constant. Then the point a has a neighborhood V and an analytic set $M \subset V$ such that

1) the equation $P_1(y, H_a^n) = 0$ can be extended to the equation $P_1(y, H_V^n) = 0$ in the neighborhood V, and the equation $P_2(z, H_b^1) = 0$ to the equation $P_2(z, H_U^1) = 0$ in a neighborhood U containing y(V);

2) Every point c which lies in the set $V \setminus M$ is regular for the equation $P_1(y, H_V^n) = 0$, and for every germ y_c which satisfies the equation $P_1(y, H_V^n) = 0$, the point y(c) is regular for the equation $P_2(z, H_U^1) = 0$;

3) for every point in the set $V \setminus M$, the ring $H_V^n\langle z_p \circ y_c\rangle$ is contained in the ring $H_V^n\langle y_c, t_c\rangle$, where y_c satisfies the equation $P_1(y, H_V^n) = 0$, $p = y(c)$, and z_p satisfies the equation $P_2(z, H_U^1) = 0$, and t_c is an analytic germ at the point c, some power of which lies in the ring $H_V^n\langle y_c\rangle$, i.e., $t^m \in H_V^n\langle y_c\rangle$.

Proof. We will assume that $y(a) = 0$. Since the equation $P_2(z, H_U^1) = 0$, it is known that $z = \varphi(\sqrt[m]{x})$, where the function φ is analytic in a neighborhood of 0. Let $P_1(y, H_a^n) = y^k + p_1 y^{k-1} + \ldots + p_k$. Consider the polynomial $Q(t, H_a^n) = t^{mk} + p_1 t^{m(k-1)} + \ldots + p_k$. Suppose that $Q(t, H_a^n) = \prod_{i=1}^{l} Q_i(t, H_a^n)$ is a decomposition of this polynomial into irreducible factors. Set $V = \bigcap_{i=1}^{l} V_i$, where V_i is the neighborhood in which the conditions of Lemma 3 hold for the equation $Q_i(t, H_a^n) = 0$ and the function φ. Let M_1 be a prohibited set for the equation $P_1(y, H_a^n) = 0$ and M_2 the set of zeros of the function $p_k(x)$. Set $M = M_1 \cup M_2$. Then the neighborhood V and the set M satisfy the conditions of the lemma. We will go into the proof of the third assertion only.

Let $c \in V \setminus M$. Then 1) the point c is regular for the equation $P_1(y, H_V^n) = 0$, since $c \in V \setminus M_1$, and 2) for every germ y_c which satisfies the equation $P_1(y, H_a^n) = 0$, $y(c) \neq 0$ since $c \in V \setminus M_2$. Since $y(c) \neq 0$, the equality $z \circ y_c = \varphi(\sqrt[m]{y_c})$ makes sense. The germ $t_c = \sqrt[m]{y_c}$ satisfies the equation $Q(t, H_a^n) = 0$ and therefore one of its irreducible equations $Q_i(t, H_a^n) = 0$. Applying Lemma 3 to the germ t_c and the function φ, we find that the ring $H_V^n\langle z_p \circ y_c\rangle$ is contained in $H_V^n\langle y_c, t_c\rangle$.

THEOREM 1. If f_a is a germ which is algebroidal in a neighborhood $U \subseteq C^n$, g_c is an algebroidal germ in a neighborhood $V \subseteq C^1$ which contains $f(U)$, and $c = f(a)$, then the germ $g_c \circ f_a$ is algebroidal in U, and for any point $b \in U$ the ring $H_b^n(g \circ f)$ is contained in the ring $RH_b^n(f)$, where $RH_b^n(f)$ is some radical extension of $H_b^n(f)$.

Proof. The first assertion of the theorem is contained in Lemma 2. Let $P(f, H_U^n) = 0$ denote the equation, irreducible in H_U^n, which is satisfied by the germ f_a. Let $\prod_{i=1}^{l} P_i(f_i, H_b^n)$ be its decomposition in the ring H^n_a and $f_i(b) = c_i$. Let $\prod_j G_{ij}(g_{ij}, H^1_{c_i})$ be the decompositions of the germs g_c into irreducible factors in the rings $H^1_{c_i}$. Set $V = \bigcap_{i,j} V_{ij}$, where the V_{ij} are neighborhoods satisfying the condition of Lemma 4 for the equations $P_i(f_i, H^n_b) = 0$ and $G_{ij}(g_{ij}, H^1_{c_i}) = 0$. Then it follows from the second assertion of Lemma 2 and from Lemma 4 that the ring $H^n_V(f \circ g)$ is contained in the ring $RH^n_V(f)$, where $RH^n_V(f)$ is some radical extension of $H^n_V(f)$. Therefore $H^n_b(g \circ f) \subseteq RH^n_b(f)$.

THEOREM 2. If $\mathbf{f}_a$ is a vector-germ which is algebroidal in a neighborhood U and g is an analytic function in a neighborhood V containing $\mathbf{f}(U)$, then the germ $g \circ \mathbf{f}_a$ is algebroidal in U, and for any point $b \in U$ the ring $H^n_b(g \circ \mathbf{f})$ is contained in $H^n_b(\mathbf{f})$.

Theorem 2 is proved using Lemmas 1 and 3, similar to the proof of Theorem 1 using Lemmas 2 and 4.

We will need the following theorem of Galois theory: a finite extension of a field of characteristic zero is radical if and only if the Galois group of the smallest Galois extension containing this finite extension is solvable.

From this theorem and Theorems 1 and 2, some corollaries follow immediately.

COROLLARY 1. In order that a function f which is algebroidal in a neighborhood U be representable in the form of superpositions of analytic functions and algebroidal functions of one variable, it is necessary that the group $\Gamma^n_a(f)$ be solvable at every point a in U.

COROLLARY 2. Let the integral irreducible equation

$$y^n + p_1 y^{n-1} + \ldots + p_n = 0,$$

where $p_i \in H^n_a$, be given in the ring H^n_a. A necessary and sufficient condition for the existence of a neighborhood U of the point a, in which an algebroidal function y satisfying this equation can be represented in the form of superpositions of analytic functions and algebroidal functions of one variable, is the solvability of the group $\Gamma^n_a(y)$.

We now go into the geometric sense of the group $\Gamma^n_U(f)$.

Let f_a be an algebroidal germ in the neighborhood U, M its prohibited set, V a neighborhood which lies in U, $c \in V \setminus M$ a regular point of the germ f_a, and $Q_c = (f_{1c}, \ldots, f_{kc})$ the complete set of germs which are equivalent to f_a on the neighborhood U at the point c. Take a closed arc $\gamma(t)$, lying in $V \setminus M$ whose initial point is c.

Continuing the germs f_{ic} along $\gamma(t)$, we get some permutation of the set Q_c. It is easy to see that the indicated correspondence gives us a homomorphism τ of the fundamental group $\pi_1(V \setminus M)$ into the group S(k) of permutations of k elements, $\tau\colon \pi_1(V \setminus M) \to S(k)$.

Definition 7. The image of the fundamental group under the homomorphism described above is called the monodromy group of the germ f_a on V, and is denoted by the symbol $M^n_V(f)$.

The monodromy group $M^n_V(f)$ acts naturally on the field $L^n_V(f) = L^n_V\langle f_{1c}, \ldots, f_{kc}\rangle$. L^n_V is the invariant field with respect to this action. Thus $M^n_V(f)$ coincides with the group $\Gamma^n_V(f)$.

If $W \subseteq U$, then $M^n_W(f) \subseteq M^n_U(f)$. Now let X be an arbitrary linearly connected subset of U, and let $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_m \supseteq \ldots \supseteq X$, $\bigcap_i U_i = X$, where the U_i are connected neighborhoods in U. Then $M^n_{U_1} \supseteq M^n_{U_2} \supseteq \ldots$. Therefore the groups $M^n_{U_i}$ stabilize.

Definition 8. The monodromy group of a germ f_a on a linearly connected set $X \subseteq C^n$ is its monodromy group on a sufficiently small connected neighborhood V of the set X. This group is denoted by the symbol $M^n_X(f)$.

It is not difficult to show that $M^n_X(f)$ is isomorphic to $\Gamma^n_X(f)$.

LEMMA 5. Consider the irreducible equation

$$y^n + p_1(x) y^{n-1} + \ldots + p_n(x) = 0$$

in a simply connected region U of the complex plane. Suppose that the function $y(x)$ does not have multiple branch points, i.e., at every point $b \in U$ there exist not fewer than $n-2$ distinct analytic germs y_{ib} which satisfy this equation. Then $\Gamma^1_U(y) = S(n)$.

<u>Proof.</u> Let $D = \{a_i\}$ denote the discriminant set. Let c be some point in $U \setminus D$ and let γ_i be a closed curve in $U \setminus D$ with initial point c which does not pass through the point a_i. The curves γ_i form a basis in the group $\pi_1(U \setminus D)$. By the hypothesis, either identity permutations or transpositions correspond to these curves in the monodromy group $M^1_U(y)$. The monodromy group is transitive, since the equation is irreducible. The unique transitive group of permutations spanned by the transpositions is the group of all permutations.

Consider the equation $z^5 + az + b = 0$ over the ring H^2_0. We will compute the group $\Gamma^2_0(z)$, where $z(a,b)$ is the algebraic function defined by this equation. $\Gamma^2_0(z)$ is isomorphic to the group $M^2_U(z)$, where U is a sufficiently small neighborhood of the origin. Consider the algebraic function $z(a_0,b)$ of the variable b for $a_0 \neq 0$. Its discriminant set consists of the four solutions of the equation $5^5 b^4 = 4^4 a_0^5$. It is easy to verify that at each point of the discriminant set exactly two roots coincide. Therefore, by Lemma 5, the monodromy group of the function $z(a_0, b)$ for $a_0 \neq 0$ is S(5). We can choose a_0 so small in absolute value that all four points of the discriminant set lie in the neighborhood U. Hence it follows that the group $\Gamma^2_0(z)$ is S(5). Since S(5) is not solvable, by Corollary 2 the algebraic function of two variables, $z(a_0, b)$, defined by the equation $z^5 + az + b = 0$, cannot be represented in the form of superpositions of analytic functions and algebroidal functions of one variable in any neighborhood of the origin.

We note that the function $z(a, b)$ can easily be represented in the form of superpositions of algebraic functions of one variable and arithmetic operations (addition, multiplication, division) on them. In particular, we have proved that the operation of division must be involved in any such representation.

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