# THE AMOEBA SOLITAIRE

THE SOLUTION

It is impossible! When I say that it is impossible, I do not mean that I do not know how to do it; I mean that we can *prove* that no matter what you try, you will never succeed. Let's see how to prove that.

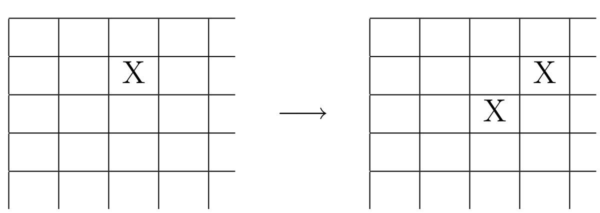
## 1. The trick

First, we are going to assign a number to each square of the chess board. This number will be called the *weight* of the square. We assign the numbers as this picture shows:

| | | | | |
|---|---|---|---|---|
| 1 | 1/2 | 1/4 | 1/8 | |
| 1/2 | 1/4 | 1/8 | 1/16 | |
| 1/4 | 1/8 | 1/16 | 1/32 | |
| 1/8 | 1/16 | 1/32 | 1/64 | |
| | | | | |

In words, the corner NW square has weight 1, each square in the next SW-NE diagonal has weight 1/2, each square in the next SW-NE diagonal has weight 1/4, then 1/8, and so on.

The reason for this weight is as follows. When an amoeba reproduces, the sum of the weights of the two daughters is the same as the weight of the mother. For instance, in the following step




each daughter has weight 1/16, and the mother has weight 1/8.

$$1/16 + 1/16 = 1/8.$$

As a consequence, *the sum of the weights of the squares where there is an amoeba is always constant.* This is the key idea. Since, at the beginning there is only one amoeba and it lies in a square with weight

1, the sum of the weights of the squares where there is an amoeba will always be 1.

As an example, say we are asked whether the following configuration of amoebas can be obtained:

| | | | | |
|---|---|---|---|---|
| | | X | | |
| | | X | | |
| X | X | | | |
| | | | | |
| | | | | |

The answer is no. The sum of their weights is

$$1/4 + 1/8 + 1/4 + 1/8 = 3/4 \neq 1.$$

There has to be some other amoeba apart from those in the picture, so that the total sum of weights is exactly 1.

## 2. The solution

Now let's go back to our original problem. We wanted to check whether we could empty the NW 3x3 corner. We are going to calculate the sum of the weights of all the squares outside of that region, and it is going to come out to less than 1. As a consequence, emptying the NW 3x3 corner will be impossible, as no matter how many amoebas we place outside of it and where, they never add up to 1. Sneaky, eh?

To do this calculation, it is easier to find first the sum of the weights of all the squares in the board. If we add them diagonal by diagonal, the result it

$$1 + 2\frac{1}{2} + 3\frac{1}{4} + 4\frac{1}{8} + 5\frac{1}{16} + \ldots = \sum_{n=0}^{\infty} \frac{n+1}{2^n}$$

This is a series whose sum we computed in class, and the value is

$$\sum_{n=0}^{\infty} \frac{n+1}{2^n} = 4$$

The sum of the weights of the squares on the NW 3x3 corner is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 3 + \frac{1}{16}$$

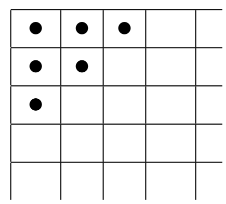
Hence the sum of the weights of all the squares outside of the NW 3x3 corner is

$$4 - \left(3 + \frac{1}{16}\right) = \frac{15}{16} < 1$$

This completes the proof.

## 3. A follow-up problem

If you liked this problem, here is another one. Can you empty the following region of size 6 (the one marked by the bullets)?



If yes, show how. If not, prove it.

## Questions?

Email me at `alfonso@math.toronto.edu`.